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Vortex motion in the spatially inhomogeneous conservative Ginzburg–Landau model

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Abstract

Using the method of matched asymptotic expansion we construct the mobility relation describing the vortex dynamics in the conservative Ginzburg–Landau equation with a spatially varying supercriticality parameter. For a particular case of a circularly symmetric supercriticality profile, we present the equations of vortex dynamics in a Hamiltonian form, and demonstrate the dynamic confinement of a vortex pair.

1. Introduction

The problem of motion of topological defects on a spatially inhomogeneous background arises in connection with studies of vortex confinement in superfluids and superconductors, and of vortical patterns in nonlinear optics. Schwarz [1] studied the motion of a quantized vortex in superfluid He^4 attracted to a pinning site on a macroscopically rough surface. The numerical study was recently extended to the motion of a pair of vortices near a pinning site. Recent numerical studies of dynamics of spiral waves in 2D demonstrated confinement of spiral vortices of like topological charge due to a spatial inhomogeneity [3]. Confinement of an ensemble of optical vortices with a non-vanishing total charge in a inhomogeneous Gaussian profile of a laser beam is a characteristic feature of vortical optical patterns in ring cavities [4,5].

In this communication, we shall consider the motion of vortices in the conservative Ginzburg–Landau (Gross–Pitaevsky) equation in 2D with a variable supercriticality parameter. The basic equation can be written as

$$-iu_t = \nabla^2 u + (e^{2\mu(r)} - |u|^2)u. \tag{1}$$

It is assumed that the relief $\mu(r)$ varies on a scale large compared to the size of the vortex core. Topologically non-trivial vortical solutions have a general form

$$u(r,t) = \rho(r,t)e^{i\theta(r,t)}, \tag{2}$$

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where the phase θ of a vortex with a topological charge $n \neq 0$ placed at some point $\boldsymbol{r} = \boldsymbol{r}_0$ satisfies the circulation condition

$$\oint_{\Gamma} \nabla\theta \cdot d\boldsymbol{l} = 2\pi n \tag{3}$$

on any closed contour Γ surrounding this point. The modulus $\rho(\boldsymbol{r}_0)$ vanishes to accommodate the phase singularity at this location; thus, the vortex position can be defined precisely as a zero of the complex order parameter field. A stationary solution corresponding to an isolated circularly symmetric vortex exists at $\mu =$ const. The vortex is set into motion when the circular symmetry is broken owing to spatial inhomogeneities, external fields, or presence of other vortices.

In the following, we shall derive the equation of vortex motion using the method of matched asymptotic expansion of the core and far field solutions [6,7]. Basic ideas and assumptions involved in this method are briefly discussed in Section 2, followed by the analysis of the defect core region in Section 3, and matching with the far field solution that describes global superflow on the inhomogeneous background in Section 4. The matching procedure results in a mobility relationship connecting the vortex velocity relative to the ambient superfluid with the local gradient of the supercriticality parameter.

The rest of the paper is dedicated to the motion in a circularly symmetric relief of μ. For this case, we shall present (Section 5) the dynamics of an ensemble of vortices in a Hamiltonian form similar to the classical Hamiltonian equations of vortex motion on a homogeneous background. In the last Section 6, we give explicit results for an exactly solvable supercriticality profile, and show that trajectories of a pair of vortices are periodic, and confined in an annular region surrounding the center of symmetry.

2. Matched asymptotic expansion

The method of matched asymptotic expansion is applicable when both the local gradient of the relief μ and the phase gradient induced by other vortices or external fields are sufficiently small, so that the superfluid velocity is much less than the speed of sound, and the supercriticality parameter does not change considerably on distances comparable with the core size. The formal expansion parameter ϵ has the meaning of the ratio of the core size, which is of $O(1)$ in basic units of Eq. (1), to a characteristic outer scale. It is understood that the latter characterizes both the distribution of vortices and the inhomogeneity scale, so that the inverse average distance between the vortices, their velocity, and $|\nabla\mu|$ are all of $O(\epsilon)$.

Different approximations have to be applied to the basic equation (1) in the vortex core region and in the far field. The core solution is obtained in a comoving coordinate frame centered on the vortex, and is assumed to be stationary in this frame. Both the vortex velocity (as yet unknown) and the local gradient of the supercriticality parameter can be considered as small perturbations causing a deviation from the zero-order circularly symmetric solution; a vortex acceleration or a non-uniformity of the gradient would appear only as higher-order effects.

The far field solution is obtained in the global ("laboratory") coordinate frame. The boundary conditions at infinity should insure vanishing of the superfluid velocity induced by any particular vortex. In the outer region, Eq. (1) is rewritten using extended spatial and temporal variables. Since the superfluid density in the outer region is determined, in the leading order, by the local values of the supercriticality parameter, only the phase equation has to be solved there. Unlike the superflow on a homogeneous background, the velocity field is not harmonic, and is induced both by the vortices and by the supercriticality gradients. Although computation of a superflow velocity field of this kind is, generally, a very complex problem, one can show (see Section 4) that the *inner* asymptotics of the far field solution near vortex locations matches correctly the *outer* asymptotics of the solution in the vortex core region.

The matching conditions allow to determine the vortex velocity as a function of the local gradient of the supercriticality parameter. Actually, the mobility relationship incorporating this dependence can be obtained with the help of the Fredholm's alternative, without solving perturbation equations. One has, however, to find the asymptotic behavior of the first-order correction to the symmetric inner solution in order to check the functional form of the matching conditions. The coincidence of the inner and outer asymptotics, that allows to match them by varying a single adjustable parameter – the vortex velocity – indicates that the entire procedure has been successful.

The method of matched asymptotic expansion outlined above has been applied earlier to the dynamics of vortices in the dissipative Ginzburg–Landau equation [6,7], and to the problem of curvature-driven motion of a vortex line in superfluid in three dimensions [8]. There is no need in this method in the theory of vortices in a homogeneous two-dimensional superfluid, since, as a consequence of Galilean invariance, the mobility relationship reduces there to a simple advection of vortices by the ambient superfluid. All factors causing relative motion with respect to the surrounding superfluid are sometimes referred to as a Magnus force [9]. Unlike advection, the Magnus force causes a deformation of the vortex core in the same order of magnitude, which necessitates application of matching methods and leads to nonlinear mobility relationships.

Application of matched asymptotic expansion encounters great difficulties beyond the leading-order approximation. In the second order in ϵ, compressibility effects should be taken into account. Due to acoustic emission by moving vortices, the energy is radiated to infinity, and the vortices drift to lower energy levels [10]. While we shall not consider this higher-order effect, one should keep in mind that the vortex confinement conditions we shall obtain are impaired at very long times, and that the approximations we use are invalidated at close approach of vortices.

3. Expansion in the core region

In the core region, we use the comoving coordinate frame centered on the vortex position $\boldsymbol{r}_0$ and propagating with a slow velocity $\epsilon\boldsymbol{v}$. Stationary solutions in the comoving frame verify the equation

$$i\epsilon\boldsymbol{v}\cdot\nabla u+\nabla^2 u+(e^{2\mu(\boldsymbol{r})}-|u|^2)u=0. \tag{4}$$

The local relief in the core region is presented by the Taylor series

$$\mu(\boldsymbol{r})=\mu(\boldsymbol{r}_0)+\epsilon\boldsymbol{M}\cdot\boldsymbol{r}+\cdots\equiv\mu_0+\epsilon\mu_1+\cdots,$$

where $\boldsymbol{M}=\nabla\mu(\boldsymbol{r}_0)$.

A weak external phase gradient is identified with the ambient superfluid velocity $\boldsymbol{v}_s=2\nabla\theta$ that advects the vortex without deformation. This velocity component can be removed by the gauge transformation $u\to u\exp(-\frac{1}{2}i\boldsymbol{v}_s\cdot\boldsymbol{r})$. The velocity entering Eq. (4) can be seen therefore as the velocity *relative* to the ambient superfluid.

We are looking for a solution of Eq. (4) in the form of an expansion in the small parameter ϵ:

$$u=u_0+\epsilon u_1+\epsilon^2 u_2+\cdots.$$

The *zero-order* solution is written in the polar coordinate frame r, ϕ as $u_0=\rho_0(r)e^{in\phi}$, where the function $\rho_0(r)$ verifies the stationary zero-order equation

$$\rho_0''+r^{-1}\rho_0'+(e^{2\mu_0}-(n/r)^2-\rho_0^2)\rho_0=0. \tag{5}$$

The parameter μ_0 can be eliminated by rescaling $\rho_0 \to e^{\mu_0}\bar{\rho}_0$, $r \to re^{-\mu_0}$. The zero-order real amplitude can be expressed therefore as $\rho_0(r) = e^{\mu_0}\bar{\rho}_0(e^{\mu_0}r)$, where the function $\bar{\rho}_0(r)$ verifies the equation

$$\bar{\rho}_0'' + r^{-1}\bar{\rho}_0' + (1 - (n/r)^2 - \bar{\rho}_0^2)\bar{\rho}_0 = 0. \tag{6}$$

The *first-order* equation reads

$$\mathcal{L}(u_1, u_1^*) = -i\mathbf{v} \cdot \nabla u_0 - 2e^{2\mu_0}\mu_1 u_0, \tag{7}$$

where $\mathcal{L}$ is the linear operator

$$\mathcal{L}(u_1, u_1^*) = \nabla^2 u_1 + (e^{2\mu_0} - 2|u_0|^2)u_1 - u_0^2 u_1^*. \tag{8}$$

The solvability condition of the first-order equation (7) is the orthogonality of the inhomogeneous part to the eigenfunction ∇u_0^*. In order to avoid divergences in the far field, the solvability condition has to be obtained by integrating over a circle with a radius $L = O(\epsilon^{-1/2}) \gg 1$, and involves both the area and the contour integrals:

$$\mathrm{Re}\left\{ \int_0^L r\,dr \int_0^{2\pi} d\phi \nabla u_0^* \left(i\mathbf{v} \cdot \nabla u_0 + 2e^{2\mu_0}\mu_1 u_0\right) + L \int_0^{2\pi} d\phi \left(\nabla u_0^* \cdot \partial_r u_1 - u_1 \cdot \partial_r \nabla u_0^*\right)_{r=L} \right\} = 0. \tag{9}$$

The contour integral depends on the asymptotics of the first-order field u_1. Since $\rho_0(L) = e^{\mu_0}(1 - O(\epsilon))$, the contour integral can be expressed, to the leading order, through the phase field only. Using the polar representation (2) we can present ρ and θ by the asymptotic expansion

$$\rho = \rho_0(r) + \epsilon\chi(r)\cos\phi, \quad \theta = n\phi + \epsilon\eta(r)\sin\phi, \tag{10}$$

where the polar angle ϕ is counted from the direction of the local gradient $\boldsymbol{M}$ of the relief of μ. The forcing term containing $M = |\boldsymbol{M}|$ in the first-order equations can be compensated by a velocity component in the normal direction, which indicates that the vortex velocity relative to the ambient superfluid is orthogonal to the direction of the relief gradient. We shall denote $v = \pm|\boldsymbol{v}|$, and consider this value to be positive when the velocity is turned anticlockwise by $\pi/2$ relative to the direction of $\boldsymbol{M}$. Then the equations for the first-order functions χ and η take the form

$$\begin{aligned} &\chi'' + r^{-1}\chi' + \left[e^{2\mu_0} - 3\rho_0^2 - (n^2+1)r^{-2}\right]\chi - 2nr^{-2}\rho_0\eta = -\rho_0(nr^{-1}v + 2re^{2\mu_0}M), \\ &\eta'' + r^{-1}\eta' - r^{-2}\eta + 2\rho_0^{-1}(\rho_0'\eta' - nr^{-2}\chi) = v\rho_0'\rho_0^{-1}. \end{aligned} \tag{11}$$

The leading terms in the asymptotics of η and χ are

$$\eta(r) \asymp nMr\ln\alpha r, \quad \chi(r) \asymp Me^{\mu_0}r, \tag{12}$$

where the parameter α is as yet indefinite. Using the expression of η, the contour integral is evaluated as

$$-2\pi n^2 e^{2\mu_0} M(\tfrac{1}{2} + \ln\alpha L).$$

The area integral in (9) is evaluated as

$$-\pi \int_0^L dr\rho_0'(r)\rho_0(r)(2nv + 2Mr^2e^{2\mu_0}).$$

The term proportional to v computed immediately as $-\pi v n e^{2\mu_0}$, while the second term can be calculated numerically using the core solution, and presented in the form $-2\pi n^2 M e^{2\mu_0} \ln(a_1 L e^{\mu_0})$, where the constant a_1 is given numerically as $\ln a_1 = 0.405$. Thus, the mobility relation takes the form

$$v = 2nM\left(\tfrac{1}{2} + \ln \frac{\alpha e^{-\mu_0}}{a_1}\right). \tag{13}$$

The parameter α will be computed by matching the asymptotic expression (10) with the far field solution in the next section. Since θ is a multivalued function, it is convenient to use for matching purposes its derivatives which can be represented as a components of the vector $\boldsymbol{U}$ defined as

$$\nabla\theta = n\mathfrak{J}\boldsymbol{U}, \quad \boldsymbol{U} = \frac{\boldsymbol{r}}{r^2} + \epsilon \boldsymbol{M} \ln \alpha e r - \epsilon \frac{\boldsymbol{r}\cdot\boldsymbol{M}}{r^2}\boldsymbol{r}, \tag{14}$$

where $\mathfrak{J}$ denotes the operator of rotation by $\pi/2$.

4. The far field solution and matching

The far field equation, which is applicable on distances large compared to the core size can be formally obtained by rescaling in Eq. (1) the coordinates $\boldsymbol{r} \to \boldsymbol{R} = \epsilon \boldsymbol{r}$ and time $t \to T = \epsilon^2 t$. In the lowest order, only the phase equation is relevant:

$$\nabla^2\theta + 2\nabla\mu \cdot \nabla\theta = 0, \tag{15}$$

where the operator ∇ refers to the extended coordinates $\boldsymbol{R} = (X, Y)$, and the phase θ should also verify the circulation condition (3). This equation can be interpreted as the continuity equation describing the flow of a superfluid with a variable density. Note that this equation, being Galilean invariant, is also valid in a moving frame (this is in contrast to the relaxative Ginzburg–Landau equation where terms proportional to the velocity appear in the lowest order phase equation).

It is convenient to introduce a dual function related to θ as follows:

$$\theta_X = -n(\Phi_Y + 2\Phi\mu_Y),$$
$$\theta_Y = n(\Phi_X + 2\Phi\mu_X). \tag{16}$$

Eq. (15) is the integrability condition for Φ, while the integrability condition for θ combined with the circulation condition (3) results in the equation of Φ containing a point source at the vortex location $\boldsymbol{R} = \boldsymbol{R}_0$:

$$\nabla \cdot [\nabla\Phi + 2\Phi\nabla\mu] = 2\pi\delta(\boldsymbol{R} - \boldsymbol{R}_0). \tag{17}$$

The vector $\boldsymbol{U}$ defined by Eq. (14) can be presented in the far field as

$$\boldsymbol{U} = \nabla\Phi + 2\Phi\nabla\mu.$$

On the other hand, Eq. (14) rewritten in the extended coordinates takes the form

$$\boldsymbol{U} = \frac{\boldsymbol{R} - \boldsymbol{R}_0}{|\boldsymbol{R} - \boldsymbol{R}_0|^2} + \nabla\mu(\boldsymbol{R}_0) \ln(\gamma e|\boldsymbol{R} - \boldsymbol{R}_0|) - \frac{(\boldsymbol{R} - \boldsymbol{R}_0)\cdot\nabla\mu(\boldsymbol{R}_0)}{|\boldsymbol{R} - \boldsymbol{R}_0|^2}(\boldsymbol{R} - \boldsymbol{R}_0), \tag{18}$$

with $\gamma = \alpha/\epsilon$. Setting $\Phi = e^{(\mu_0 - \mu)}H$ reduces Eq. (17) to the form

$$\nabla^2 H - \left[|\nabla\mu|^2 - \nabla^2\mu\right] H = 2\pi\delta(\boldsymbol{R} - \boldsymbol{R}_0). \tag{19}$$

The required asymptotics (18) for the vector $\boldsymbol{U}$ is obtained if $H(\boldsymbol{R}, \boldsymbol{R}_0) \asymp \ln(e\gamma|\boldsymbol{R} - \boldsymbol{R}_0|)$ at $\boldsymbol{R} \to \boldsymbol{R}_0$.

Using the Taylor expansion for $\mu(\boldsymbol{R})$ at $\boldsymbol{R}=\boldsymbol{R}_0$ one can see that Eq. (19) reduces asymptotically at $\boldsymbol{R}\to\boldsymbol{R}_0$ to

$$\nabla^2 H - m^2 H = 2\pi\delta(\boldsymbol{R}-\boldsymbol{R}_0), \tag{20}$$

where

$$m^2 = |\nabla\mu(\boldsymbol{R}_0)|^2 - \nabla^2\mu(\boldsymbol{R}_0). \tag{21}$$

The solution of Eq. (20) is expressed through the modified Bessel function,

$$H = -K_0(m|\boldsymbol{R}-\boldsymbol{R}_0|)\,, \tag{22}$$

with the asymptotic behavior $H \asymp \ln(e^C m|\boldsymbol{R}-\boldsymbol{R}_0|/2)$, where C is the Euler constant $C=0.577\ldots$. Comparing this expression with the outer asymptotic (14) of the core solution gives the value of the parameter α in the mobility relation (13), $\alpha = \frac{1}{2}\epsilon m e^{C-1}$. The final form of the mobility relation defining the vortex velocity relative to the ambient superfluid is

$$\boldsymbol{v} = -2n\mathfrak{J}\nabla\mu(\boldsymbol{r}_0)\ln\left(\frac{\epsilon m e^{C-1/2-\mu_0}}{2a_1}\right). \tag{23}$$

5. Hamiltonian equations of motion

It is evident that an isolated vortex moves along a line $\mu=$ const [1]. When more than one vortex is present, the superflow induced by each vortex on the inhomogeneous background can be obtained using the far field solution (22).

Consider the system of N vortices with topological charges n_i located at the points $\boldsymbol{r}_i$ in the relief $\mu(\boldsymbol{r})$. The velocity of ith vortex is determined as the sum of the total superfluid velocity induced by all other vortices at this location, which is given by the components of the vector $\boldsymbol{U}$, and the velocity of relative motion due to the local relief. Denoting the superflow in the location point of the ith vortex as $\boldsymbol{A}_i$, we obtain

$$\begin{aligned}\boldsymbol{A}_i &= \sum_{k\neq i}^{N} n_k\nabla\theta_k(\boldsymbol{r}_i) = \sum_{k\neq i}^{N} n_k\mathfrak{J}\boldsymbol{U}_k(\boldsymbol{r}_i) = \mathfrak{J}e^{-2\mu_i}\sum_{k\neq i}^{N} n_k\nabla[e^{2\mu_i}\Phi(\boldsymbol{r}_i,\boldsymbol{r}_k)] \\ &= \mathfrak{J}e^{-2\mu_i}\sum_{k\neq i}^{N} n_k\nabla[e^{\mu_i+\mu_k}H(\boldsymbol{r}_i,\boldsymbol{r}_k)] = -\mathfrak{J}e^{-2\mu_i}\sum_{k\neq i}^{N} n_k\nabla[e^{\mu_i+\mu_k}K_0(m_i|\boldsymbol{r}_i-\boldsymbol{r}_k|)],\end{aligned} \tag{24}$$

where $\theta_k(\boldsymbol{r}_i)$ is the phase field induced by the kth vortex; $\mu_i \equiv \mu(\boldsymbol{r}_i)$ and $m_i = \left[|\nabla\mu(\boldsymbol{r}_i)|^2 - \nabla^2\mu(\boldsymbol{r}_i)\right]^{1/2}$.

In the following, we shall restrict ourselves to a monotonous circularly symmetric relief $\mu(r)$. In this case, r and, consequently, $B(r)=\ln m(r)$ can be expressed as a function of μ only. Then the velocity $\boldsymbol{v}_i$ of relative motion of the ith vortex due to the local relief is expressed as

$$\begin{aligned}\boldsymbol{v}_i &= -2n_i\mathfrak{J}\nabla\mu_i[B(\mu_i) - \mu_i + \ln(\epsilon e^{C-1/2}/2a_1)] \\ &= -\mathfrak{J}e^{-2\mu_i}n_i\nabla[(\ln G - \mu_i)e^{2\mu_i} + 2\int^{\mu_i} d\mu e^{2\mu}B(\mu)],\end{aligned} \tag{25}$$

[1] This behavior has been, indeed, confirmed by direct numerical simulation of Eq. (1) [11]. With more vortices, computation is very time-consuming, and sensitive to radiation effects.

where $G = \epsilon e^C/2a_1$.

Using this expression together with (24) we present the equations of motion in the Hamiltonian form:

$$n_i e^{2\mu(r_i)} \dot{\boldsymbol{r}}_i = \mathcal{J}\frac{\partial \mathcal{H}}{\partial \boldsymbol{r}_i}. \quad (26)$$

The conserved Hamiltonian is

$$\mathcal{H} = \sum_i^N \left(F_i + \sum_{k\neq i}^N V_{ik} \right).$$

The function F_i describes the action of the relief on the ith vortex, while V_{ik} represents the advective interaction of the respective pair of vortices:

$$F_i(r_i) = -n_i^2 \left\{ e^{2\mu(r_i)}[\ln G - \mu(r_i)] + 2\int^{\mu(r_i)} d\mu e^{2\mu} B(\mu) \right\}, \quad (27)$$

$$V_{ik}(\boldsymbol{r}_i, \boldsymbol{r}_k) = -n_i n_k e^{\mu(r_i)+\mu(r_k)} K_0(\exp[B(\mu(r_i))]|\boldsymbol{r}_i - \boldsymbol{r}_k|), \quad (28)$$

For some simple profiles of μ, the integral in (27) can be expressed in a closed form. For example, for a general power relief function $\mu(r) = -ar^n$,

$$B(\mu) = -(1/n)\ln(-\mu/a) + (1/2)\ln[n^2\mu(\mu-1)],$$

and the function F_i takes the form

$$F_i = -n_i^2 e^{2\mu_i} \left\{ (1/2)\ln\left[G^2 n(-\mu_i/a)^{-1/n}\sqrt{\mu_i(\mu_i - 1)}\right] - \mu_i \right\} - n_i^2 \left[\frac{2-n}{4n}\mathrm{Ei}(2\mu_i) - \frac{e^2}{4}\mathrm{Ei}(2\mu_i - 2) \right], \quad (29)$$

where $\mathrm{Ei}(x)$ is the exponential integral.

In addition to the energy invariant $\mathcal{H}$, Eq. (26) has an angular momentum invariant stemming from the circular symmetry of the relief. Multiplying the equation of motion (26) by $\boldsymbol{r}_i$, and taking note of orthogonality condition $(\boldsymbol{r}_i \cdot \mathcal{J}\boldsymbol{r}_i) = 0$, we obtain

$$(1/2) n_i e^{2\mu(r_i)} \dot{r}_i^2 = -2\sum_{k\neq i}^N n_i n_k e^{\mu_i+\mu_k} \exp[B(\mu_i)] K_1(\exp[B(\mu_i)]|\boldsymbol{r}_i - \boldsymbol{r}_k|) \frac{\boldsymbol{r}_i \cdot \mathcal{J}\boldsymbol{r}_k}{|\boldsymbol{r}_i - \boldsymbol{r}_k|}.$$

Summing up similar expressions over all i and taking into account the obvious relation

$$\boldsymbol{r}_i \cdot \mathcal{J}\boldsymbol{r}_k + \boldsymbol{r}_k \cdot \mathcal{J}\boldsymbol{r}_i = 0$$

we see that the sum vanishes. Therefore the system has the integral of motion

$$\mathcal{P} \equiv \sum_i^N n_i e^{2\mu(r_i)} r_i^2 = \text{const}. \quad (30)$$

This is the generalization of a known invariant for the system of point vortices in a flat relief $\mu = 0$ [12].

It is important to distinguish between two qualitatively different cases. When all n_i are of the same sign, e.g. $n_i > 0$, assuming that $0 < e^{2\mu(r)} \leq 1$, one can show that the rotational integral (30) is topologically a sphere. Thus, the dynamics unfolds on a $(2N-2)$-dimensional manifold, and the phase space, considered as the intersection of integrals in R^{2N}, is compact. The motion of vortices is confined to this sphere. An inhomogeneous relief leads therefore to the *dynamical confinement* of vortices. The second case corresponds to n_i with different signs; the phase space is now non-compact, and trajectories may escape to infinity or, on the contrary, collapse in a point. An example in the next section shows, however, that dynamical confinement is possible also in this case.

6. Bessel relief

It is interesting to find some non-trivial relief functions $\mu(\boldsymbol{r})$ which allow to obtain an exact solution of Eq. (19). This is possible when, for example, m = const in the entire plane. To make this condition more transparent, we substitute $\mu = -\ln \Omega$. Then the function Ω should verify the equation

$$\nabla^2 \Omega = m^2 \Omega. \tag{31}$$

The simplest relief function satisfying this condition is $\mu(\boldsymbol{r}) = \boldsymbol{m} \cdot \boldsymbol{r}$, where $\boldsymbol{m}$ is a constant vector. A more interesting case, which we shall consider in detail, is a circularly symmetric function $\Omega(\boldsymbol{r})$. Placing the origin of the coordinate system in the center of symmetry, we obtain the relief function

$$\mu(r) = -\ln I_0(mr). \tag{32}$$

The supercriticality profile $e^{2\mu(r)}$ ("Bessel relief") looks qualitatively similar to a Gaussian relief (Fig. 1). Using Eq. (22) yields the function Φ:

$$\Phi(\boldsymbol{r}, \boldsymbol{r}_0) = -\frac{I_0(mr)}{I_0(mr_0)} K_0(m|\boldsymbol{r} - \boldsymbol{r}_0|).$$

The mobility relation (23) becomes

$$v = 2nm\hat{\jmath}\boldsymbol{r} \frac{I_1(mr)}{rI_0(mr)} \ln\left[\frac{\epsilon m e^{C-1/2}}{2a_1} I_0(mr)\right]. \tag{33}$$

The Hamiltonian form of the equation of motion (26) is preserved. The functions describing the influence of the relief on an isolated vortex as well as the mutual interaction of vortices have the following form:

$$\begin{aligned} F_i(\boldsymbol{r}_i) &= -n_i^2 e^{2\mu(r_i)} [\ln(Gm) - \mu(r_i)], \\ V_{ik}(\boldsymbol{r}_i, \boldsymbol{r}_k) &= -n_i n_k e^{\mu(r_i)+\mu(r_k)} K_0(m|\boldsymbol{r}_i - \boldsymbol{r}_k|), \end{aligned} \tag{34}$$

In a particular case of two vortices, the Hamiltonian depends on three variables, and its level surfaces are two-dimensional. Due to the existence of another integral of motion, Eq. (30), we can expect, at most, periodic motion. The true character of motion is seen most clearly in the three-dimensional phase space spanned by the radii of the two vortices r_1, r_2 and their mutual distance r_{12}. Figs. 2, 3 show a one-dimensional *confinement manifold* obtained as the intersection of the surfaces $\mathcal{P}$ = const and $\mathcal{H}$ = const. Both in cases of likely and unlikely charged vortices, this manifold is topologically a *finite* line segment. The actual trajectory must lie within this segment, although it does not necessarily cover it entirely.

An example of a confinement manifold in the case of likely charged vortices, that consists of two symmetric branches, is shown in Fig. 2. The periodic motion of the vortices is projected on the three-dimensional phase space as shuttling along this segment. Confinement of unlikely charged vortices does not follow immediately

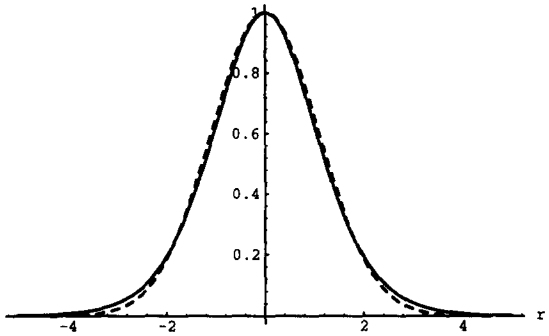

Fig. 1.

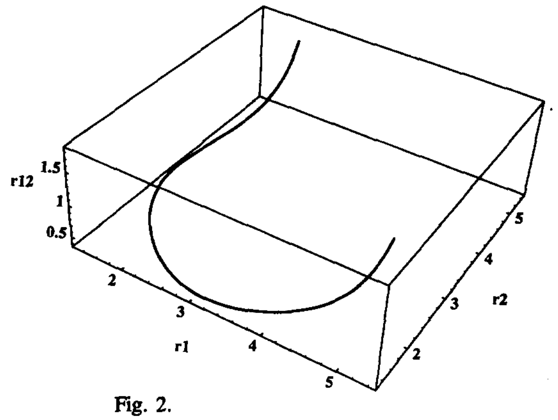

Fig. 2.

Fig. 1. The Bessel relief $I_0^{-2}(r)$ (solid line) compared to the Gaussian relief $\exp(-0.65^2r^2)$ (dashed line).

Fig. 2. The confinement manifold for likely charged vortices ($\mathcal{H} = 1.938$, $\mathcal{P} = 4.782$).

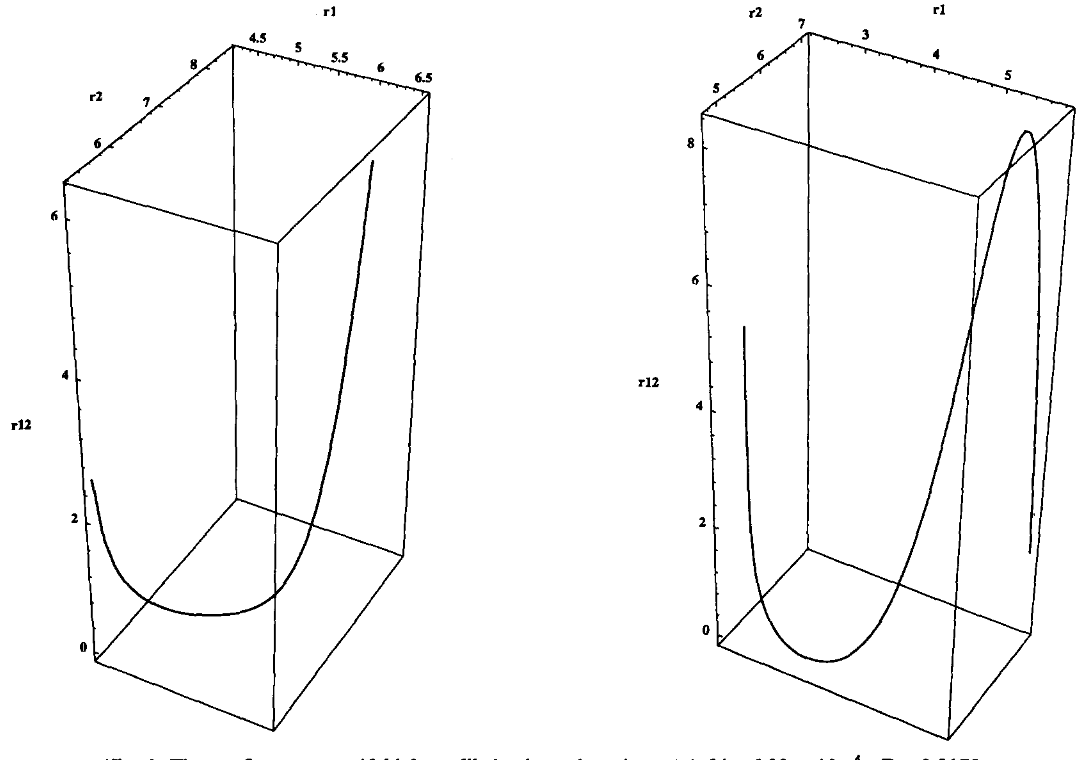

Fig. 3. The confinement manifold for unlikely charged vortices. (a) $\mathcal{H} = 6.29 \times 10^{-4}$, $\mathcal{P} = 0.0178$.

from Eq. (30). Nevertheless, computations show that a finite confinement manifold exists also in this case. Fig. 3 shows two examples of a confinement manifold for a pair of vortices with opposite charges; note a peculiar shape of the manifold in Figs. 3b. Evidently, the motion is finite and periodic also in this case.

The confinement is apt to be impaired in the next approximation when the compressibility of the superfluid is taken into account. This effect is very weak at low Mach numbers corresponding to the case of well separated vortices and small supercriticality gradients considered above. The confinement manifolds for a pair of vortices shown in Fig. 3 exhibit, however, a region of close approach where r_{12} is small. In this region, the vortices accelerate and the radiation enhances. At very close approach, the long-scale approximation breaks down, and annihilation accompanied by a burst of radiation is likely.

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