Sylvester-Cayley vector partitions algorithm and the Gaussian polynomials

Boris Y. Rubinstein
Stowers Institute for Medical Research
1000 50th St., Kansas City, MO 64110, U.S.A.

December 15, 2021

Abstract

We extend an algorithm suggested in 1858 by Sylvester [13] and implemented in 1860 by Cayley [4] for a problem of double partitions and apply it to derivation of explicit expressions for coefficients of the Gaussian polynomials through convolution of restricted partition functions.

Keywords: double partitions, Gaussian polynomials, Bernoulli polynomials of higher order, Eulerian numbers.

2010 Mathematics Subject Classification: 11P82.

1 Restricted partition functions

1.1 Restricted scalar partition function

Consider a problem of finding a number of nonnegative integer solutions of a Diophantine equation

$$\sum_{i=1}^{m} x_i d_i = \mathbf{x}\cdot\mathbf{d} = s, \quad \mathbf{d} = \{d_1, d_2, \dots, d_m\}. \tag{1}$$

A restricted partition function $W(s,\mathbf{d})$ solving the above problem is a number of partitions of an integer s into positive integers $\{d_1, d_2, \dots, d_m\}$, each not greater than s. The generating function for $W(s,\mathbf{d})$ has a form

$$G(t,\mathbf{d}) = \prod_{i=1}^{m} \frac{1}{1-t^{d_i}} = \sum_{s=0}^{\infty} W(s,\mathbf{d})\, t^s\ , \tag{2}$$

In other words the partition function equals a constant term in Taylor expansion of the following expression

$$W(s,\mathbf{d}) = \mathrm{const}_t \left[t^{-s} \prod_{i=1}^{m} (1-t^{d_i})^{-1} \right]. \tag{3}$$

Sylvester proved [14] a statement about splitting of the partition function into periodic and non-periodic parts and showed that the restricted partition function may be presented as a sum of the *Sylvester waves* [11]

$$W(s,\mathbf{d}) = \sum_{j=1} W_j(s,\mathbf{d})\ , \tag{4}$$

where summation runs over all distinct factors j of the elements d_i. The wave $W_j(s,\mathbf{d})$ is a quasipolynomial in s closely related to prime roots ρ_j of unity; it is a coefficient of t^{-1} in the series expansion in ascending powers of t of the generator

$$F_j(s,t)=\sum_{\rho_j}\frac{\rho_j^{-s}e^{st}}{\prod_{k=1}^{m}\left(1-\rho_j^{d_k}e^{-d_k t}\right)}\ . \tag{5}$$

The summation is made over all prime roots of unity $\rho_j=\exp(2\pi in/j)$ for n relatively prime to j (including unity) and smaller than j. It was shown [11] that it is possible to express the Sylvester wave as a finite sum of the Bernoulli polynomials of higher order.

1.2 Restricted vector partition function

Consider a function $W(\mathbf{s},\mathbf{D})$ counting the number of integer nonnegative solutions $\mathbf{x}\geq 0$ to the linear system $\mathbf{D}\cdot\mathbf{x}=\mathbf{s}$, where $\mathbf{D}$ is a nonnegative integer $l\times m$ matrix. The *vector partition function* (VPF) $W(\mathbf{s},\mathbf{D})$ is a natural generalization of the restricted partition function to the vector argument.

The generating function for the VPF reads

$$G(\mathbf{t},\mathbf{D})=\prod_{i=1}^{m}\frac{1}{1-\mathbf{t}^{\mathbf{c}_i}}=\sum_{\mathbf{s}}W(\mathbf{s},\mathbf{D})\mathbf{t}^{\mathbf{s}},\quad \mathbf{t}^{\mathbf{s}}=\prod_{k=1}^{l}t_k^{s_k},\quad \mathbf{t}^{\mathbf{c}_i}=\prod_{k=1}^{l}t_k^{d_{ki}}, \tag{6}$$

where $\mathbf{c}_i,\ 1\leq i\leq m$, denotes the i-th column of the matrix $\mathbf{D}$. Note that some elements d_{ki} might equal zero.

The partition problems that include inequalities can be reduced to the VPF problem as follows. Assume that we have a system of k linear equations and $l-k$ linear inequalities for $m>l$ integer nonnegative variables x_i:

$$\sum_{i=1}^{m}d_{ji}x_i=s_j,\ (1\leq j\leq k),\quad \sum_{i=1}^{m}d_{ji}x_i\leq n_j,\ (k+1\leq j\leq l). \tag{7}$$

Introduce $l-k$ new integer variables $x_i,\ (k+1\leq i\leq l)$ that enter additively all above equations and inequalities with corresponding factors $d_{ji}=0$ for $1\leq j\leq k$, and $d_{ji}=1$ for $k+1\leq j\leq l$, and transform all inequalities into equations

$$\sum_{i=1}^{m+l-k}d_{ji}x_i=s_j,\ (1\leq j\leq k),\quad \sum_{i=1}^{m+l-k}d_{ji}x_i=n_j,\ (k+1\leq j\leq l). \tag{8}$$

It is easy to see that the number of integer nonnegative solutions of (8) equals that of (7), and thus any system of linear equations with constraints can be reduced to the standard vector partition problem.

1.3 Sylvester-Cayley method of compound partitions

The problem of vector partitions has a long history and J.J. Sylvester made a significant contribution to its solution. Sylvester [13] wrote: *"Any given system of simultaneous simple equations to be solved in positive integers being proposed, the determination of the number of solutions of which they admit may in all cases be made to depend upon the like determination for one or more systems of equations of a certain fixed standard form. When a system of r equations between n variables of the aforesaid* *standard form is given, the determination of the number of solutions in positive integers of which it admits may be made to depend on the like determination for*

$$\frac{n(n-1)\dots(n-r+2)}{1\cdot 2\dots(r-1)}$$

single **independent** *equations derived from those of the given system by the ordinary process of elimination, with a slight modification; the final result being obtained by taking the sum of certain numerical multiples (some positive, others negative) of the numbers corresponding to those independent determinations. This process admits of being applied in a variety of modes, the resulting sum of course remaining unaltered in value whichever mode is employed, only appearing for each such mode made up of a different set of component parts."*

Then he added in the footnote: *"If there be r simultaneous simple equations between n variables (in which the coefficients are all positive or negative integers) forming a definite system (that is, one in which no variable can become indefinitely great in the positive direction without one or more of the others becoming negative), and if the r coefficients belonging to each of the same variable are exempt from a factor common to them all, and if not more than $r-1$ of the variables can be eliminated simultaneously between the r equations, then the determination of the number of positive integer solutions of the given system may be made to depend on like determinations for each of n derived independent systems, in each of which the number of variables and equations is one less than in the original system."*

In other words, Sylvester claimed that VPF can be reduced to a sum of $\binom{n}{r-1}$ scalar partition functions (see also [9]) and the reduction is an iterative process based on the variable elimination. Sylvester considered a specific double partition problem as an illustration of his method and determined regions (*chambers*) on a plane $\{s_1, s_2\}$ each having a unique expression for VPF valid in this region only. He showed that the expressions in the adjacent chambers coincide at their common boundary (see also [12]).

This approach was successfully applied by Cayley [4] to double partitions subject to some restrictions on the *positive* elements of matrix $\mathbf{D}$ (the columns $\mathbf{c}_i$ are linearly independent and for all $1 \le i \le m$ the inequality $c_{i2} < s_2 + 2$ holds). Cayley noticed: *"The subject (as I am aware) has hardly been considered except by Professor Sylvester, and it is greatly been regretted that only an outline of his valuable researches has been published: the present paper contains the demonstration of a theorem, due to him, by which (subject to certain restrictions) the question of Double Partitions is made to depend upon the ordinary theory of Single Partitions"*.

In this manuscript we present the result for double partitions obtained by Cayley and modify it to apply to the Gaussian polynomial problem. A discussion of the Sylvester-Cayley method to multiple partitions with $l > 2$ will be considered elsewhere.

2 Double partitions

Introduce an augmented matrix $\mathbf{E}$ obtained by prepending the column vector $\mathbf{s}$ to the matrix $\mathbf{D}$, so that $e_{10} = s_1 = r$ and $e_{20} = s_2 = \rho$. Denote the rest elements of first row of $\mathbf{E}$ as $e_{1i} = d_{1i} = b_i$, $(1 \le i \le m)$, and of the second row by $e_{2i} = d_{2i} = \beta_i$, $(1 \le i \le m)$. For simplicity assume that all $\beta_i > 0$, and perform a partial fraction expansion (PFE) step to present $G(\mathbf{t}, \mathbf{D})$ as sum of m fractions

$$G(\mathbf{t}, \mathbf{D})\mathbf{t}^{-\mathbf{s}} = \mathbf{t}^{-\mathbf{s}} \prod_{i=1}^{m} \frac{1}{1 - t_1^{b_i} t_2^{\beta_i}} = \sum_{i=1}^{m} T_i(\mathbf{t}), \quad T_i = \frac{A_i(\mathbf{t})\mathbf{t}^{-\mathbf{s}}}{1 - t_1^{b_i} t_2^{\beta_i}}. \tag{9}$$

Then the solution reads as a sum of the terms

$$U_i = \mathrm{const}_{\mathbf{t}}\left[T_i(\mathbf{t})\right]. \tag{10}$$

Cayley showed that the first term U_1 corresponding to elimination of the second column of matrix $\mathbf{E}$ can be written as

$$U_1 = \mathrm{const}_{t_1}\left[t_1^{-(r\beta_1 - b_1\rho)}\prod_{j\neq 1}^{m}(1 - t_1^{b_j\beta_1 - b_1\beta_j})^{-1}\right]. \tag{11}$$

This result can be obtained using a simple transformation of the original system of equations $\mathbf{D}\cdot\mathbf{x} = \mathbf{s}$. Namely, divide the second equation by $\beta_1 \neq 0$, so that $\bar{e}_{2i} = \bar{d}_{2i} = \beta_i/\beta_1,\ (1 \le i \le m),\ \bar{e}_{20} = \rho/\beta_1$. Denote $t_2 = a$, perform PFE to

$$t_1^{-r}a^{-\rho/\beta_1}\prod_{i=1}^{m}(1 - t_1^{b_i}a^{\beta_i/\beta_1})^{-1} = \sum_{i=1}^{m}\tau_i(\mathbf{t}),$$

and find that τ_1 corresponds to a simple pole $a_1 = t_1^{-b_1}$, and is equal to

$$\tau_1 = \frac{t_1^{-r}a^{-\rho/\beta_1}}{(a - a_1)B'(a_1)}, \quad B(a) = \prod_{j=1}^{m}(1 - t_1^{b_j}a^{\beta_j/\beta_1}). \tag{12}$$

The derivative evaluates to

$$B'(a) = -\sum_{i=1}^{m}t_1^{b_i}(\beta_i/\beta_1)a^{\beta_i/\beta_1 - 1}\prod_{j\neq i}^{m}(1 - t_1^{b_j}a^{\beta_j/\beta_1}).$$

Using $a = a_1 = t_1^{-b_1}$ find the only surviving term (for $i = 1$)

$$B'(a_1) = -t_1^{b_1}\prod_{j\neq 1}^{m}(1 - t_1^{b_j - b_1\beta_j/\beta_1}),$$

and

$$\tau_1 = -t_1^{-r-b_1}\frac{a^{-\rho/\beta_1}}{(a - t_1^{-b_1})}\prod_{j\neq 1}^{m}(1 - t_1^{b_j - b_1\beta_j/\beta_1})^{-1}. \tag{13}$$

The constant term w.r.t. a can be evaluated as the residue of τ_1/a at $a = 0$, so that we have

$$\mathrm{Res}(a^{-\rho/\beta_1 - 1}/(a - t_1^{-b_1}), a = 0) = -t_1^{(\rho/\beta_1 + 1)b_1},$$

and finally arrive at

$$\tau_1(t_1) = \frac{1}{t_1^{r - b_1\rho/\beta_1}}\prod_{j\neq 1}^{m}(1 - t_1^{b_j - b_1\beta_j/\beta_1})^{-1}. \tag{14}$$

The contribution U_1 is found as constant term in expansion of $\tau_1(t_1)$ in Taylor series w.r.t. t_1, and it can be checked by direct computation that

$$U_1 = \mathrm{const}_{t_1}\left[t_1^{-(r\beta_1 - b_1\rho)}\prod_{j\neq 1}^{m}(1 - t_1^{b_j\beta_1 - b_1\beta_j})^{-1}\right]. \tag{15}$$

Comparing this to (3) we see that U_1 corresponds to $W(r\beta_1 - b_1\rho, \mathbf{d}_1)$, where the elements d_{1j} of the vector $\mathbf{d}_1$ are given by $d_{1j} = b_j\beta_1 - b_1\beta_j,\ j \neq 1$. Effectively, this result corresponds to elimination [6] of the first unknown x_1 in the system $\mathbf{D} \cdot \mathbf{x} = \mathbf{s}$ that leads to a single Diophantine equation $\mathbf{d}_1 \cdot \mathbf{x}' = r\beta_1 - b_1\rho$, where $\mathbf{x}'$ is the vector of unknowns obtained from $\mathbf{x}$ by dropping x_1.

Similar transformations allow to find all m contributions to obtain

$$W(\mathbf{s}, \mathbf{D}) = \sum_{i=1}^{m} U_i = \sum_{i=1}^{m} W(L_i, \mathbf{d}_i), \quad L_i = r\beta_i - b_i\rho, \quad d_{ij} = b_j\beta_i - b_i\beta_j,\ j \neq i, \tag{16}$$

where both L_i and d_{ij} as the determinants of 2×2 matrices made of the columns $\{\mathbf{c}_0,\ \mathbf{c}_i\}$ and $\{\mathbf{c}_j,\ \mathbf{c}_i\}$, respectively. Cayley mentioned [4] that the contribution of each term in (16) is nonzero only when L_i is nonnegative, thus reintroducing the notion of *chambers* of the vector partition. He also pointed out that when some elements of the vector $\mathbf{d}_i$ are negative (say, $d_{ij_k} < 0$ for $1 \le k \le K$) one has

$$W(L_i, \mathbf{d}_i) = (-1)^K W(L_i - \sum_{k=1}^{K} |d_{ij_k}|, |\mathbf{d}_i|), \quad |\mathbf{d}_i| = \{|d_{ij}|\}. \tag{17}$$

2.1 Matrix with zero elements in one row

Consider now a case when one or more elements in one (say, the second) row of the matrix $\mathbf{D}$ are zeroes. Denote the number of nonzero elements as m_0, and without loss of generality we assume that $\beta_i \neq 0$ for $1 \le i \le m_0$, and $\beta_i = 0$ for $m_0 + 1 \le i \le m$. Similar to (9) we have

$$G(\mathbf{t}, \mathbf{D})\mathbf{t}^{-\mathbf{s}} = \mathbf{t}^{-\mathbf{s}} \prod_{i=1}^{m_0} (1 - t_1^{b_i} t_2^{\beta_i})^{-1} \prod_{i=m_0+1}^{m} (1 - t_1^{b_i})^{-1} = \sum_{i=1}^{m_0} T_i(\mathbf{t}), \tag{18}$$

the number m_0 of the terms T_i is equal to the number of the partial fractions generated by PFE. Again in order to obtain the term T_1 introduce $\bar{e}_{2i} = \bar{d}_{2i} = \beta_i/\beta_1,\ (1 \le i \le m),\ \bar{e}_{20} = \rho/\beta_1$. Perform PFE to

$$t_1^{-r} a^{-\rho/\beta_1} \prod_{i=m_0+1}^{m} (1 - t_1^{b_i})^{-1} \prod_{i=1}^{m_0} (1 - t_1^{b_i} a^{\beta_i/\beta_1})^{-1} = \sum_{i=j}^{m_0} \tau_j(\mathbf{t}),$$

and find that τ_j corresponds to a simple pole $a_1 = t_1^{-b_1}$, and is equal to

$$\tau_1 = \frac{t_1^{-r} a^{-\rho/\beta_1}}{(a - a_1)B'(a_1)} \prod_{i=m_0+1}^{m} (1 - t_1^{b_i})^{-1}\ , \quad B(a) = \prod_{j=1}^{m_0} (1 - t_1^{b_j} a^{\beta_j/\beta_1})\ . \tag{19}$$

Using $a = a_1 = t_1^{-b_1}$ obtain

$$B'(a_1) = -t_1^{b_i} \prod_{j \neq 1}^{m_0} (1 - t_1^{b_j - b_1\beta_j/\beta_1})\ ,$$

and

$$\tau_1 = -t_1^{-r-b_1} \frac{a^{-\rho/\beta_1}}{(a - a_1)} \prod_{j \neq 1}^{m_0} (1 - t_1^{b_j - b_1\beta_j/\beta_1})^{-1} \prod_{i=m_0+1}^{m} (1 - t_1^{b_i})^{-1}\ . \tag{20}$$

Recalling that $\beta_i = 0$ for $m_0 + 1 \le i \le m$ rewrite expression (refmCayley50) in more compact form

$$\tau_1 = -t_1^{-r-b_1} \frac{a^{-\rho/\beta_1}}{(a - a_1)} \prod_{j \neq 1}^{m} (1 - t_1^{b_j - b_1\beta_j/\beta_1})^{-1}\ , \tag{21}$$

which coincides with (13). This means that the result (16) remains valid in this case too, but the number of terms in this sum reduces to m_0. The elimination in this case is performed for all columns with nonzero second element and we obtain

$$W(\mathbf{s},\mathbf{D}) = \sum_{i=1}^{m_0} W(L_i,\mathbf{d}_i), \quad L_i = d_{i0} = r\beta_i - b_i\rho, \quad d_{ij} = b_j\beta_i - b_i\beta_j,\ j \neq i. \tag{22}$$

It should be noted that determinants L_i can be computed for all $1 \le i \le m$, but for $m_0+1 \le i \le m$ their values are always negative, so that the corresponding partition functions vanish and thus do not contribute into the final result.

2.2 Matrix with zero elements in both rows

The above algorithm does not work when zeros appear in both rows and we have to use a different approach. Sort the matrix columns to have $b_i = 0$ for $1 \le i \le n < m$ and produce an auxiliary matrix $\mathbf{D}'$ made of the last $m-n$ columns of the matrix $\mathbf{D}$. The generating function $G(\mathbf{t},\mathbf{D})$ reads

$$G(\mathbf{t},\mathbf{D}) = G(\mathbf{t},\mathbf{D}')\prod_{i=1}^{n}\frac{1}{1-t_2^{\beta_i}}. \tag{23}$$

Assume that the partition $W(\mathbf{s},\mathbf{D}')$ is known. In case $n=1$ the product in (23) degenerates into a single factor $1/(1-t_2^{\beta_1})$ that can be presented as an infinite series $\sum_{k_1} t_2^{k_1\beta_1}$. This leads to

$$\sum_{\mathbf{s}} W(\mathbf{s},\mathbf{D})\mathbf{t}^{\mathbf{s}} = \sum_{k_1=0}^{\infty}\sum_{\mathbf{s}} W(\mathbf{s},\mathbf{D}')\mathbf{t}^{\mathbf{s}} t_2^{k_1\beta_1},$$

which in its turn produces a finite series for $\mathbf{s} = \{s_1, s_2\}$

$$W(\mathbf{s},\mathbf{D}) = \sum_{k_1=0}^{\lfloor s_2/\beta_1\rfloor} W(\mathbf{s}-k_1\beta_1\hat{\mathbf{e}}_2,\mathbf{D}'), \tag{24}$$

where $\lfloor x\rfloor$ denotes the largest integer smaller or equal to x and $\hat{\mathbf{e}}_2 = \{0,1\}$ is a unit vector in direction s_2. The finite number of terms in (24) are due to the requirement of nonnegativeness of all components of the vector argument of the partition function. For $n > 1$ the relation (24) generalizes to

$$W(\mathbf{s},\mathbf{D}) = \sum_{k_1=0}^{\lfloor s_2/\beta_1\rfloor}\sum_{k_2=0}^{\lfloor s_2/\beta_2\rfloor}\ldots\sum_{k_n=0}^{\lfloor s_2/\beta_n\rfloor} W(\mathbf{s}-\hat{\mathbf{e}}_2\sum_{i=1}^{n}k_i\beta_i,\mathbf{D}'). \tag{25}$$

Thus the partition function $W(\mathbf{s},\mathbf{D})$ in this case is constructed using the following algorithm − first all columns $\mathbf{c}_i$ with first zero elements are dropped to produce the matrix $\mathbf{D}'$ for which the auxiliary partition $W(\mathbf{s},\mathbf{D}')$ is computed; it is then used in (25) to construct $W(\mathbf{s},\mathbf{D})$ as a finite sum of known partitions. The chambers of $W(\mathbf{s},\mathbf{D})$ coincide with that of $W(\mathbf{s},\mathbf{D}')$.

3 Gaussian polynomials and their coefficients

The Gaussian polynomial coefficients $P_m^n(s)$ are defined as a number of nonnegative integer solutions of a linear Diophantine equation with a constrain

$$\sum_{i=1}^{m} ix_i = s, \quad \sum_{i=1}^{m} x_i \le n. \tag{26}$$

The coefficients $P_m^n(s)$ of the Gaussian polynomial $G_m^n(t)$ satisfy [8, 15]

$$G_m^n(t) = \frac{G_m(t)G_n(t)}{G_{m+n}(t)} = \sum_{s=0}^{mn} P_m^n(s)t^s, \quad G_m(t) = \prod_{i=1}^{m} \frac{1}{1-t^i}, \tag{27}$$

where $G_m(t)$ denotes the generating function for the restricted partition function $W_m(s) = W(s, \hat{\mathbf{d}}^m = \{1, 2, \ldots, m\})$. The quasipolynomials $P_m^n(s)$ have a finite order mn and the following properties [15]

$$\begin{array}{l} P_m^n(0) = P_m^n(mn) = 1, \quad P_m^n(s) = 0, \quad \text{for} \quad s > mn, \\ P_m^n(s) = P_n^m(s) = P_m^n(mn - s), \quad P_m^n(mn/2 - s) = P_m^n(mn/2 + s), \\ P_m^n(s) - P_n^m(s-1) \geq 0, \quad \text{for} \quad 0 \leq s \leq mn/2. \end{array} \tag{28}$$

Cayley [3] considered a problem of computation of the Gaussian polynomial coefficients and found the generating functions (27) and (30, see below). Some interesting results about explicit expression of $P_m^n(s)$ through the regular partition functions were reported recently in [7].

3.1 Sylvester-Cayley algorithm for Gaussian polynomial coefficients

The problem (26) is a particular case of (7) for $k = 1,\ l = 2$, and $d_{1i} = i,\ d_{2i} = 1$, thus it can be transformed into a VPF problem

$$\sum_{i=0}^{m} i x_i = s, \quad \sum_{i=0}^{m} x_i = n, \tag{29}$$

and dealt with using the Sylvester-Cayley algorithm.

Write (22) for

$$\mathbf{D} = \left(\begin{array}{ccccc} 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & \ldots & m \end{array} \right), \quad \mathbf{s} = \left(\begin{array}{c} n \\ s \end{array} \right),$$

that determines a generating function for $P_m^n(s)$ with fixed integers m and n

$$\prod_{i=0}^{m} \frac{1}{1 - at^i} = \sum_{n=0}^{\infty} G_m^n(t)a^n = \sum_{n=0}^{\infty}\sum_{s=0}^{mn} P_m^n(s)a^n t^s. \tag{30}$$

In (22) we have $m_0 = m,\ r = n,\ \rho = s,\ b_i = 1, \beta_i = i$ and obtain $L_i = ni - s$ and $d_{ij} = \beta_i - \beta_j = i - j$. Thus the set $\mathbf{d}_i$ can be written as a union of $\hat{\mathbf{d}}^i$ and $-\hat{\mathbf{d}}^{m-i}$. We then observe from (22)

$$P_m^n(s) = \sum_{i=1}^{m} w_i(m, n; s), \quad w_i(m, n; s) = W(ni - s, -\hat{\mathbf{d}}^{m-i} \cup \hat{\mathbf{d}}^i), \tag{31}$$

where the i-th term in the above sum contributes for $0 \leq s \leq ni$. This immediately leads to a conclusion that $P_m^n(s) = 0$ for $s > nm$ as all m terms in (31) vanish. By the representation (31) the Gaussian polynomial $P_m^n(s)$ has m chambers bounded by lines $s = ni,\ 0 \leq i \leq m$. Specifically, in the k-th chamber bounded by the lines $s = n(m-k)$ and $s = n(m-k+1)$ one has to retain only k terms in the sum with $m - k + 1 \leq i \leq m$. Using (17) we arrive at

$$w_i(m, n; s) = (-1)^{m-i} W(ni - s - s_{m-i}, \hat{\mathbf{d}}^i \cup \hat{\mathbf{d}}^{m-i}), \quad s_m = m(m+1)/2, \tag{32}$$

where the partition function W can be computed using the algorithm discussed in [11]. It is convenient to introduce $P_m^n(r, s)$ that explicitly accounts for the summands entering $P_m^n(s)$ in the r-th chamber

$$P_m^n(r, s) = \sum_{i=r}^{m} w_i(m, n; s), \quad (r-1)n \leq s \leq rn. \tag{33}$$

3.2 Convolution of restricted partitions

Consider a general term $W(s, \hat{\mathbf{d}}^{k_1} \cup \hat{\mathbf{d}}^{k_2})$ in (32). Its generating function reads $G(t) = G_{k_1}(t)G_{k_1}(t)$ and we obtain

$$\sum_{s=0} W(s, \hat{\mathbf{d}}^{k_1} \cup \hat{\mathbf{d}}^{k_2}) t^s = \sum_{s_1=0}\sum_{s_2=0} W_{k_1}(s_1)W_{k_2}(s_1)t^{s_1+s_2},$$

leading to a discrete convolution (the Cauchy product)

$$W(s, \hat{\mathbf{d}}^{k_1} \cup \hat{\mathbf{d}}^{k_2}) = \sum_{k=0}^{s} W_{k_1}(k)W_{k_2}(s-k),$$

and setting here $k_1 = i,\ k_2 = m - i$ we find

$$W(s, \hat{\mathbf{d}}^{i} \cup \hat{\mathbf{d}}^{m-i}) = \sum_{k=0}^{s} W_i(k)W_{m-i}(s-k).$$

Substituting it in (32) and (31) we obtain

$$P_m^n(s) = \sum_{i=1}^{m} \sum_{k=0}^{ni-s-s_{m-i}} (-1)^{m-i} W_i(k)W_{m-i}(ni - s - s_{m-i} - k),$$

where the chamber boundaries remain unchanged. Noticing that $W_0(s) = \delta_{s,0}$ we rewrite the above expression as

$$P_m^n(s) = W_m(mn-s) + \sum_{i=1}^{m-1} \sum_{k=0}^{ni-s-s_{m-i}} (-1)^{m-i} W_i(k)W_{m-i}(ni - s - s_{m-i} - k), \tag{34}$$

where the first term in r.h.s. contributes to *all* chambers. The partial expression $P_m^n(r,s)$ for the r-th chamber reads

$$P_m^n(r,s) = W_m(mn-s) + \sum_{i=r}^{m-1} \sum_{k=0}^{ni-s-s_{m-i}} (-1)^{m-i} W_i(k)W_{m-i}(ni-s-s_{m-i}-k),\ (r-1)n \le s \le rn. \tag{35}$$

The relations (28) imply a symmetry $s \leftrightarrow mn - s$ leading to an equivalent representation

$$P_m^n(r,s) = W_m(s) + \sum_{i=i}^{r-1} \sum_{k=0}^{s-in-s_i} (-1)^{i} W_{m-i}(k)W_i(s - in - s_i - k),\ (r-1)n \le s \le rn.$$

The formulas (34,35) present a closed form of the Gaussian polynomial coefficients as a superposition of the restricted partitions discrete convolutions. Below we illustrate the above approach by deriving explicit expressions of $P_m^n(s)$ for small $m = 3, 4$ (see also [5]). More cumbersome expressions required for derivation of $P_5^n(s)$ and $P_6^n(s)$ are presented in Appendix.

3.3 Explicit formula for $P_3^n(s)$

Use (34) to find

$$P_3^n(s) = W_3(3n-s) + \sum_{i=1}^{2} \sum_{k=0}^{ni-s-s_{3-i}} (-1)^{i+1} W_i(k)W_{3-i}(ni - s - s_{3-i} - k),$$

and we have $W_1(s)=1,\ s_2=3,\ \ s_1=1$ to obtain

$$\begin{aligned} P_3^n(1,s) &= W_3(3n-s)+\sum_{k=0}^{n-s-3} W_2(k)-\sum_{k=0}^{2n-s-1} W_2(k), \quad 0\le s\le n, \\ P_3^n(2,s) &= W_3(3n-s)-\sum_{k=0}^{2n-s-1} W_2(k), \quad n\le s\le 2n, \\ P_3^n(3,s) &= W_3(3n-s), \quad 2n\le s\le 3n. \end{aligned} \tag{36}$$

Recall that $W_2(s)=s/2+3/4+(-1)^s/4$ and find

$$\Sigma_2(s)=\sum_{k=0}^{s} W_2(k)=\frac{(s+1)(s+3)}{4}+\frac{1+\cos\pi s}{8},$$

to arrive at the final expression

$$P_3^n(s)=\left\{\begin{array}{ll} W_3(3n-s)+\Sigma_2(n-s-3)-\Sigma_2(2n-s-1), & 0\le s\le n, \\ W_3(3n-s)-\Sigma_2(2n-s-1), & n\le s\le 2n, \\ W_3(3n-s), & 2n\le s\le 3n, \end{array}\right. \tag{37}$$

where

$$W_3(s)=\frac{47}{72}+\frac{s}{2}+\frac{s^2}{12}+\frac{1}{8}\cos\pi s+\frac{2}{9}\cos\frac{2\pi s}{3}. \tag{38}$$

Note that in [5] the authors completely describe $P_3^n(s)$ using 36 explicit expressions while our representation requires only three formulas in (37). It indicates that the approach based on Sylvester-Cayley algorithm employing the idea of chambers provides more compact and clear result.

3.4 Explicit formula for $P_4^n(s)$

Start with

$$P_4^n(s)=W_4(4n-s)+\sum_{i=1}^{3}\sum_{k=0}^{ni-s-s_{4-i}}(-1)^{i+1}W_i(k)W_{4-i}(ni-s-s_{4-i}-k),$$

and use $s_3=6$ to write

$$P_4^n(s)=\sum_{k=0}^{n-s-6} W_3(n-s-6-k)-\sum_{k=0}^{2n-s-3} W_2(k)W_2(2n-s-k-3)+\sum_{k=0}^{3n-s-1} W_3(k)+W_4(4n-s), \tag{39}$$

where

$$W_4(s)=\frac{2s^3+30s^2+135s+175}{288}+\frac{(s+5)}{32}\cos\pi s+\frac{1}{8}\cos\frac{\pi s}{2}+\frac{2}{27}\left(\cos\frac{2\pi s}{3}-\cos\frac{2\pi(s+1)}{3}\right). \tag{40}$$

3.5 Maximal coefficient of Gaussian polynomial

The problem of computation of the maximal coefficient of Gaussian polynomial was addressed recently in [5] (also L. Fel, private communication).

First note that by (28) the maximal coefficient p_m^n is given by $p_m^n=P_m^n(mn/2)$ for even mn and $p_m^n=P_m^n((mn+1)/2)$ for odd mn. For even $m=2k$ we have to compute $P_m^n(s)$ at $s=kn$ that belongs to the $(k+1)$-th chamber where $kn \le s \le (k+1)n$. For odd $m=2k-1$ the argument of P_m^n belongs to the middle k-th chamber with $(k-1)n \le s \le kn$. Thus we have three cases

$$p_m^n = \begin{cases} P_{2k}^n(k+1,kn), & m=2k,\\ P_{2k-1}^{2r}(k,2kr-r), & m=2k-1,\ n=2r,\\ P_{2k-1}^{2r-1}(k,2kr-k-r+1), & m=2k-1,\ n=2r-1. \end{cases} \tag{41}$$

Use here $k=2$ and for $m=3$ we find $p_3^{2r}=P_3^{2r}(2,3r)$, and $p_3^{2r-1}=P_3^{2r-1}(2,3r-1)$; with $m=4$ we obtain $p_4^n = P_4^n(3,2n)$.

Consider first the case $m=3$. For even $n=2r$ we obtain $s=3r$, so that $3n-s=3r$ and $2n-s-1=r-1$ leading to $P_3^{2r}(2,3r)=W_3(3r)-\Sigma_2(r-1)$. For odd $n=2r-1$ we use $s=3r-1$, that gives $3n-s=3r-2$ and $2n-s-1=r-1$ producing $P_3^{2r}(2,3r-1)=W_3(3r-2)-\Sigma_2(r-1)$. After algebraic transformations we arrive at

$$p_3^{2r}=\frac{1}{2}\left[(r+1)^2+\cos\frac{\pi r}{2}\right],\quad p_3^{2r-1}=\frac{r(r+1)}{2}. \tag{42}$$

For $m=4$ we find

$$P_4^n(3,2n)=W_4(2n)+\sum_{k=0}^{n-1}W_3(k),$$

that reduces to

$$p_4^n=\frac{(2n+5)^3}{288}+\frac{(2n+5)}{32}+\frac{3}{16}\cos\pi n+\frac{4}{27}\cos\frac{2\pi n}{3}+\frac{4}{27}\sin\frac{\pi(8n+1)}{6}. \tag{43}$$

Turning to case $m=5$ we use $k=3$ and consider two separate cases – for even $n=2r$ we obtain $s=5r$, so that $p_5^{2r}=P_5^{2r}(3,5r)$, while with odd $n=2r-1$ we have $p_5^{2r-1}=P_5^{2r-1}(3,5r-2)$. The corresponding expressions read

$$\begin{aligned} p_5^{2r} &= \frac{(23r^2+69r+49)(r+1)(r+2)}{288}+\frac{425}{1728}+\frac{3(2r+3)}{64}\cos\pi r+\frac{4}{27}\cos\frac{2\pi r}{3}\\ &+ \frac{1}{8}\left(\cos\frac{\pi r}{2}+\sin\frac{\pi r}{2}\right), \end{aligned} \tag{44}$$

$$\begin{aligned} p_5^{2r-1} &= \frac{(23r^2+46r+36)r(r+2)}{288}-\frac{37}{1728}-\frac{1}{64}\cos\pi r-\frac{1}{8}\sin\frac{\pi r}{2}\\ &+ \frac{2}{27}\cos\frac{2\pi r}{3}-\frac{2}{27}\sin\frac{\pi(8r+1)}{6}. \end{aligned} \tag{45}$$

Finally, for $m=6$ we again take $k=3$ to obtain $p_6^n=P_6^n(4,3n)$ leading to

$$\begin{aligned} p_6^n &= \frac{(2n+7)(66n^4+924n^3+4606n^2+9604n+12061)}{172800}+\frac{2n^2+14n+55}{256}\cos\pi n\\ &+ \frac{1}{16}\left(\cos\frac{\pi r}{2}-\sin\frac{\pi r}{2}\right)+\frac{4}{81}\cos\frac{2\pi n}{3}+\frac{4}{81}\sin\frac{\pi(4n+1)}{6}+\frac{8}{125}\left(\cos\frac{2\pi n}{5}+\cos\frac{4\pi n}{5}\right)\\ &+ \frac{4}{125}\left(-\cos\frac{\pi(2n+1)}{5}+\cos\frac{\pi(4n+1)}{5}+2\cos\frac{\pi(8n+1)}{5}\right)\\ &+ \frac{4}{125}\left(\sin\frac{\pi(4n+1)}{10}+2\sin\frac{\pi(8n+1)}{10}-\sin\frac{\pi(12n+1)}{10}\right). \end{aligned} \tag{46}$$

It is easy to see from (42-46) that the behavior of the maximal coefficient is mainly determined by the polynomial part of the corresponding expressions. This observation leads us to consideration of the polynomial part of Gaussian polynomial coefficients.

4 Polynomial part of Gaussian polynomial coefficients

4.1 Polynomial part of partition function

It is known that a leading contribution to the partition function $W(s,\mathbf{d})$ for the set of generators $\mathbf{d}=\{d_1,d_2,\ldots,d_m\}$ is provided by its polynomial part $w(s,\mathbf{d})$ that can be expressed through the Bernoulli polynomials of higher order $B_k(s,\mathbf{d})$ [11]

$$w(s,\mathbf{d})=\frac{B_{m-1}(s+\sigma_m,\mathbf{d})}{(m-1)!\pi_m},\quad \sigma_m=\sum_{i=1}^m d_i,\quad \pi_m=\prod_{i=1}^m d_i, \tag{47}$$

where the Bernoulli polynomials of higher order can be defined in umbral calculus notation as

$$B_k(s,\mathbf{d})=(s+\sum_{i=1}^m {}^iBd_i)^k,$$

after the expansion the replacement $({}^iBd_i)^k\to B_kd_i^k$ is applied, and B_k denotes the k-th Bernoulli number. This definition implies

$$B_k(s,\mathbf{d})=\sum_{l=0}^k\binom{k}{l}s^lB_{k-l}(\mathbf{d}), \tag{48}$$

and $B_k(\mathbf{d})\equiv B_k(0,\mathbf{d})$ denotes the Bernoulli number of higher order.

4.2 Polynomial part of Gaussian polynomial coefficients

It is instructive to consider a polynomial part $\mathcal{P}_m^n(s)$ of quasipolynomial $P_m^n(s)$

$$\mathcal{P}_m^n(s)=\sum_{i=1}^m\mathcal{W}_i(m,n;s), \tag{49}$$

where $\mathcal{W}_i(m,n;s)$ denotes the polynomial part of $w_i(m,n;s)$ defined in (32). For $\hat{\mathbf{d}}^m$ we have $\sigma_m=s_m,\pi_m=m!$ and we find that

$$w(s,\hat{\mathbf{d}}^i\cup\hat{\mathbf{d}}^{m-i})=\frac{B_{m-1}(s+s_i+s_{m-i},\hat{\mathbf{d}}^i\cup\hat{\mathbf{d}}^{m-i})}{(m-1)!i!(m-i)!}.$$

Use here the replacement $s\to ni-s-s_{m-i}$ to obtain

$$\mathcal{P}_m^n(s)=\sum_{i=1}^m(-1)^{m-i}\frac{B_{m-1}(ni-s+s_i,\hat{\mathbf{d}}^i\cup\hat{\mathbf{d}}^{m-i})}{(m-1)!i!(m-i)!}. \tag{50}$$

The polynomial part $\mathcal{P}_m^n(r,s)$ of the solution in r-th chamber reads

$$\mathcal{P}_m^n(r,s)=\sum_{i=r}^m(-1)^{m-i}\frac{B_{m-1}(ni-s+s_i,\hat{\mathbf{d}}^i\cup\hat{\mathbf{d}}^{m-i})}{(m-1)!i!(m-i)!},\quad (r-1)n\le s\le rn. \tag{51}$$

5 Polynomial part of maximal coefficient

The last result leads to expressions for the polynomial part of the maximal coefficient of the Gaussian polynomial. For small $m \le 6$ we have

$$\begin{aligned} &\mathcal{P}_2^n(2,n) = \frac{2n+3}{4}, \\ &\mathcal{P}_3^{2r}(2,3r) = \frac{(r+1)^2}{2} + \frac{1}{36}, \quad \mathcal{P}_3^{2r-1}(2,3r-1) = \frac{(3r+1)(3r+2)}{18}, \\ &\mathcal{P}_4^n(3,2n) = \frac{(2n+5)^3}{288}, \\ &\mathcal{P}_5^{2r}(3,5r) = \frac{(r+1)(r+2)(23r^2+69r+49)}{288} + \frac{571}{14400}, \\ &\mathcal{P}_5^{2r-1}(3,5r-2) = \frac{(r+2)r(23r^2+46r+36)}{288} + \frac{89}{1600}, \\ &\mathcal{P}_6^n(4,3n) = \frac{(2n+7)(66n^4+924n^3+4606n^2+9604n+7511)}{172800}. \end{aligned} \tag{52}$$

5.1 Polynomial part for even m

For even $m = 2k$ the maximal coefficient equals

$$\begin{aligned} \mathcal{P}_{2k}^n(k+1,kn) &= \sum_{i=k+1}^{2k} (-1)^i \frac{B_{2k-1}(ni-nk+s_i, \hat{\mathbf{d}}^i \cup \hat{\mathbf{d}}^{2k-i})}{(2k-1)!i!(2k-i)!} \\ &= \sum_{j=0}^{k} (-1)^j \frac{B_{2k-1}(n(k-j)+s_{2k-j}, \hat{\mathbf{d}}^j \cup \hat{\mathbf{d}}^{2k-j})}{(2k-1)!j!(2k-j)!} \\ &= \sum_{j=0}^{k} (-1)^j \binom{2k}{j} \frac{B_{2k-1}(n(k-j), -\hat{\mathbf{d}}^{2k-j} \cup \hat{\mathbf{d}}^j)}{(2k-1)!(2k)!}, \end{aligned} \tag{53}$$

where we employ the relation $B_n(s,-\mathbf{d}) = B_n(s+\sigma_m,\mathbf{d})$. Use here (48) to write

$$\mathcal{P}_{2k}^n(k+1,kn) = \frac{1}{(2k-1)!(2k)!} \sum_{p=0}^{2k-1} \binom{2k-1}{p} n^p \sum_{j=0}^{k} (-1)^j \binom{2k}{j} B_{2k-p-1}(-\hat{\mathbf{d}}^{2k-j} \cup \hat{\mathbf{d}}^j)(k-j)^p. \tag{54}$$

5.2 Polynomial part for odd $m = 2k-1$ and even $n = 2r$

In this case the maximal coefficient reads

$$\begin{aligned} \mathcal{P}_{2k-1}^{2r}(k,(2k-1)r) &= \sum_{i=k}^{2k-1} (-1)^{i+1} \binom{2k-1}{i} \frac{B_{2k-2}(2r(i-k)+r, -\hat{\mathbf{d}}^i \cup \hat{\mathbf{d}}^{2k-1-i})}{(2k-2)!(2k-1)!} \\ &= \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} \frac{B_{2k-2}(2r(k-1-j)+r, -\hat{\mathbf{d}}^{2k-1-j} \cup \hat{\mathbf{d}}^j)}{(2k-2)!(2k-1)!}, \end{aligned} \tag{55}$$

and we find with (48)

$$\mathcal{P}_{2k-1}^{2r}(k,(2k-1)r) = \frac{1}{(2k-2)!(2k-1)!} \sum_{p=0}^{2k-2} \binom{2k-2}{p} r^p$$

$$\times \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} B_{2k-p-2}(-\hat{\mathbf{d}}^{2k-1-j} \cup \hat{\mathbf{d}}^j)(2(k-1-j)+1)^p. \tag{56}$$

5.3 Polynomial part for odd $m = 2k-1$ and $n = 2r-1$

We observe

$$\begin{aligned}
\mathcal{P}_{2k-1}^{2r-1}(k, (2k-1)r-k+1) &= \sum_{i=k}^{2k-1} (-1)^{i+1} \binom{2k-1}{i} \frac{B_{2k-2}((2r-1)(i-k)+r-1, -\hat{\mathbf{d}}^i \cup \hat{\mathbf{d}}^{2k-1-i})}{(2k-2)!(2k-1)!} \\
&= \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} \frac{B_{2k-2}((2r-1)(k-1-j)+r-1, -\hat{\mathbf{d}}^{2k-1-j} \cup \hat{\mathbf{d}}^j)}{(2k-2)!(2k-1)!}, \\
&= \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} \frac{B_{2k-2}(2r(k-1-j)+r+(j-k), -\hat{\mathbf{d}}^{2k-1-j} \cup \hat{\mathbf{d}}^j)}{(2k-2)!(2k-1)!},
\end{aligned}$$

leading to

$$\mathcal{P}_{2k-1}^{2r-1}(k, (2k-1)r-k+1) = \frac{1}{(2k-2)!(2k-1)!} \sum_{p=0}^{2k-2} \binom{2k-2}{p} r^p$$

$$\times \sum_{j=0}^{k-1} (-1)^j \binom{2k-1}{j} B_{2k-p-2}(j-k, -\hat{\mathbf{d}}^{2k-1-j} \cup \hat{\mathbf{d}}^j)(2(k-1-j)+1)^p. \tag{57}$$

6 Leading term of maximal coefficient

Consider derivation of a general expression for the leading term of the maximal coefficient. The leading in n term in the summand in (53) evaluates to

$$(-1)^j n^{2k-1} \binom{2k}{j} \frac{(k-j)^{2k-1}}{(2k-1)!(2k)!},$$

so that the leading term $L_{n,k}$ of $\mathcal{P}_{2k}^n(k+1, kn)$ reads

$$L_{kn} = \frac{n^{2k-1}}{(2k-1)!(2k)!} \sum_{j=0}^{k} (-1)^j \binom{2k}{j} (k-j)^{2k-1}, \tag{58}$$

where the sum evaluates to $A_{2k-1,k}$ being a particular case of the Eulerian numbers of type A [2]

$$A_{n,k} = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n. \tag{59}$$

These numbers used in a definition of the polylogarithm function

$$Li_{-n}(\rho) = \sum_{k=1}^{\infty} k^n \rho^k = \frac{1}{(1-\rho)^{n+1}} \sum_{i=0}^{n} A_{n,i} \rho^{n-i}, \quad \rho \neq 1,$$

which in its turn relates to the Eulerian polynomials $H_n(\rho)$ defined through the generating function

$$\frac{1-\rho}{e^t-\rho}=\sum_{n=0}^{\infty}H_n(\rho)\frac{t^n}{n!},\quad H_n(\rho)=(-1)^n\frac{1-\rho}{\rho}Li_{-n}(\rho).$$

It is worth to note that the function $H_n(\rho)$ can be extended into the Eulerian polynomials of higher order used in the derivation [11] of the partition function $W(s,\mathbf{d}^m)$. From (58, 59) it follows that for even m the leading term L_{mn} of $P_m^n(m/2+1,mn/2)$ reads

$$L_{mn}=\frac{n^{m-1}A_{m-1,m/2}}{(m-1)!m!},$$

producing $L_{2n}=n/2,\ L_{4n}=n^3/36,\ L_{6n}=11n^5/14400$ that coincide with the leading term of the corresponding expressions in (52).

In case of odd $m=2k-1$ we observe from (52) that $L_{2k-1,2r-1}=L_{2k-1,2r}$ (it is also can be seen by comparing (56) to (57)), so that it is sufficient to consider the case of even $n=2r$ only. Using (56) we obtain

$$L_{2k-1,2r}=\frac{r^{2k-2}}{(2k-2)!(2k-1)!}\sum_{j=0}^{k-1}(-1)^j\binom{2k-1}{j}(2(k-1-j)+1)^{2k-2}. \tag{60}$$

The sum in (60) evaluates to the central MacMahon numbers $B_{2k-1,k}$ being a particular case of the Eulerian numbers $B_{n,k}$ of type B that satisfy [1] (also see Ch.11 in [10])

$$B_{n,k}=(2(n-k)+1)B_{n-1,k-1}+(2k+1)B_{n-1,k},\quad B_{n,k}=\sum_{j=0}^{k}(-1)^{k-j}\binom{n+1}{k-j}(2j+1)^n,$$

and we find

$$L_{2k-1,2r}=\frac{r^{2k-2}B_{2k-1,k}}{(2k-2)!(2k-1)!},\quad\rightarrow\quad L_{mn}=\frac{(n/2)^{m-1}B_{m,(m+1)/2}}{(m-1)!m!},$$

producing $L_{3,2r}=r^2/2,\ L_{5,2r}=23r^4/288$ coinciding with the leading term of the corresponding expressions in (52).

Collecting the above results we obtain a general expression for the leading term of the maximal coefficient of the Gaussian polynomial through the Eulerian numbers

$$L_{mn}=\frac{1}{(m-1)!m!}\begin{cases}n^{m-1}A_{m-1,m/2}, & \text{even } m,\\ (n/2)^{m-1}B_{m,(m+1)/2}, & \text{odd } m,\ \text{even } n,\\ ((n+1)/2)^{m-1}B_{m,(m+1)/2}, & \text{odd } m,n.\end{cases} \tag{61}$$

Acknowledgments

The author thanks L.Fel for drawing attention to the Gaussian polynomials problem and numerous fruitful discussions.

A Computation of $P_5^n(s)$ and $P_6^n(s)$

As it follows from (31) and (32) the solution for $m=5$ can be constructed using only three basic expressions. Namely, we have

$$\begin{aligned}
W(s,\hat{\mathbf{d}}^2\cup\hat{\mathbf{d}}^3) &= \frac{s^4+18s^3+112s^2+279s+220}{288}+\frac{37}{1728}+\frac{2s+9}{64}(-1)^s+\frac{2}{27}\cos\frac{2\pi s}{3},\\
W(s,\hat{\mathbf{d}}^1\cup\hat{\mathbf{d}}^4) &= \frac{s^4+22s^3+166s^2+495s+465}{576}+\frac{25}{3456}+\frac{2s+11}{128}(-1)^s\\
&+ \frac{2}{27}\sin\frac{(4s+1)\pi}{6}+\frac{1}{16}\left(\cos\frac{\pi s}{2}+\sin\frac{\pi s}{2}\right),\\
W(s,\hat{\mathbf{d}}^5) &= \frac{s^4+30s^3+310s^2+1275s+1687}{2880}+\frac{41}{86400}+\frac{2s+15}{128}(-1)^s+\frac{2}{27}\cos\frac{2\pi s}{3}\\
&+ +\frac{1}{16}\left(\cos\frac{\pi s}{2}+\sin\frac{\pi s}{2}\right)+\frac{2}{25}\left(\cos\frac{2\pi s}{5}+\cos\frac{4\pi s}{5}\right). \qquad \text{(A1)}
\end{aligned}$$

Then we use (32) to find $w_i(5,n;s)$ to insert into (31) or (33) for explicit expression of $P_5^n(s)$ or $P_5^n(r,s)$ respectively.

The solution for $m=6$ is produced based on the four following expressions

$$\begin{aligned}
W(s,\hat{\mathbf{d}}^3\cup\hat{\mathbf{d}}^3) &= \frac{6s^5+180s^4+2020s^3+10440s^2+24299s+19650}{25920}+\frac{s+6}{64}(-1)^s\\
&+ \frac{6s+34}{243}\cos\frac{2\pi s}{3}+\frac{4}{243}\sin\frac{\pi(4s+1)}{6},\\
W(s,\hat{\mathbf{d}}^2\cup\hat{\mathbf{d}}^4) &= \frac{12s^5+390s^4+4720s^3+26130s^2+64458s+54275}{69120}+\frac{2s^2+26s+75}{512}(-1)^s\\
&+ \frac{2}{81}\cos\frac{2\pi s}{3}+\frac{2}{81}\sin\frac{\pi(4s+1)}{6}+\frac{1}{32}\left(\cos\frac{\pi s}{2}+\sin\frac{\pi s}{2}\right),\\
W(s,\hat{\mathbf{d}}^1\cup\hat{\mathbf{d}}^5) &= \frac{6s^5+240s^4+3560s^3+24000s^2+71325s+70888}{86400}+\frac{s+8}{128}(-1)^s\\
&+ \frac{2}{81}\cos\frac{2\pi s}{3}+\frac{2}{81}\sin\frac{\pi(4s+1)}{6}+\frac{1}{16}\sin\frac{\pi s}{2}+\frac{2}{125}\left(\cos\frac{2\pi s}{5}+\cos\frac{4\pi s}{5}\right)\\
&+ \frac{2}{125}\left(\cos\frac{\pi(2s+1)}{5}+3\cos\frac{\pi(6s+1)}{5}+2\cos\frac{\pi(8s+1)}{5}\right)\\
&- \frac{2}{125}\left(2\sin\frac{\pi(8s+1)}{10}+\sin\frac{\pi(12s+1)}{10}+3\sin\frac{\pi(16s+1)}{10}\right),\\
W(s,\hat{\mathbf{d}}^6) &= \frac{12s^5+630s^4+12320s^3+110250s^2+439810s+598731}{1036800}+\frac{6s^2+126s+581}{4608}(-1)^s\\
&+ \frac{6s+61}{486}\cos\frac{2\pi s}{3}+\frac{2}{243}\sin\frac{\pi(4s+1)}{6}+\frac{1}{18}\cos\frac{\pi s}{3}\\
&+ \frac{1}{32}\left(\cos\frac{\pi s}{2}+\sin\frac{\pi s}{2}\right)+\frac{2}{125}\left(\cos\frac{2\pi s}{5}+\cos\frac{4\pi s}{5}\right)\\
&+ \frac{2}{125}\left(\cos\frac{\pi(2s+1)}{5}+3\cos\frac{\pi(6s+1)}{5}+2\cos\frac{\pi(8s+1)}{5}\right)\\
&- \frac{2}{125}\left(2\sin\frac{\pi(8s+1)}{10}+\sin\frac{\pi(12s+1)}{10}+3\sin\frac{\pi(16s+1)}{10}\right). \qquad \text{(A2)}
\end{aligned}$$

Now employ (32) to obtain $w_i(6,n;s)$ and then use it in (31) or (33) for explicit expression of $P_6^n(s)$ or $P_6^n(r,s)$ respectively.

References

[1] A. Borowiec and W. Mlotkowski, *New Eulerian Numbers of Type D*, arXiv:1509.03758v1 [math.CO].

[2] L. Carlitz, *Eulerian Numbers and Polynomials*, Mathematics Magazine, **32** (1959), 247-260.

[3] A. Cayley, *Researches on the Partition of Numbers*, Phil. Trans. Roy.l Soc. London, **CXLV** (1855), 127-140; Coll. Math. Papers, **II** (1891), 235-249.

[4] A. Cayley, *On a Problem of Double Partitions*, Philosophical Magazine, **XX** (1860), 337-341; Coll. Math. Papers, **IV** (1891), 166-170.

[5] A. Castillo, S. Flores, A. Hernandez, B. Kronholm, A. Larsen and A. Martinez, *Quasipolynomials and Maximal Coefficients of Gaussian Polynomials*, Ann. Comb., **15** (2019), 589-611.

[6] L.E. Dickson, *History of the Theory of Numbers*, **II**, (1920), Ch. III, 123.

[7] L.G. Fel, *Gaussian Polynomials and Restricted Partition Function with Constraints*, arXiv:1611.09931v1 [math.CO].

[8] C.F. Gauss, *Quarumdam Serierum Singularium*, Comm. Soc. Reg. Sci. Gottingen Rec., **1** (1811); Werke, **II**, (1863), 16-21.

[9] G.B. Mathews, *On the Partition of Numbers*, Proc. London Math. Soc., **28** (1896-97), 486-490.

[10] K.T. Petersen, *Eulerian Numbers*, Birkhauser (2015).

[11] B.Y. Rubinstein, *Expression for Restricted Partition Function through Bernoulli Polynomials*, Ramanujan Journal, **15** (2008), 177-185.

[12] B. Sturmfels, *Note on Vector Partition Functions*, J. Combinat. Theory, **A72** (1995), 302-309.

[13] J. J. Sylvester, *On the Problem of the Virgins, and the General Theory of Compound Partitions*, Philosophical Magazine, **XVI** (1858), 371-376; Coll. Math. Papers, **II** (1908), 113-117.

[14] J. J. Sylvester, *On Subinvariants, i.e. Semi-invariants to Binary Quantics of an Unlimited Order. With an Excursus on Rational Fractions and Partitions*, American Journal of Mathematics, **V** (1882), 79-136; Coll. Math. Papers, **III** (1909), 568-622.

[15] J. J. Sylvester, *A Constructive Theory of Partitions, Arraged in Three Acts, an Interact and an Exodion*, American Journal of Mathematics, **V** (1882), 251-330; **VI** (1884), 334-336; Coll. Math. Papers, **IV** (1912), 1-83.