Power Sums Related to Semigroups $S\,(d_1, d_2, d_3)$

Leonid G. Fel[†] and Boris Y. Rubinstein[‡]

[†]Department of Civil and Environmental Engineering,

Technion, Haifa 32000, Israel

and

[‡]Department of Mathematics, University of California, Davis,

One Shields Ave., Davis, CA 95616, U.S.A.

September 26, 2005

Abstract

The explicit formulas for the sums of positive powers of the integers s_i unrepresentable by the triple of integers $d_1, d_2, d_3 \in \mathbb{N},\ \gcd(d_1, d_2, d_3) = 1$, are derived.

Let $\mathsf{S}\,(d_1, \ldots, d_m)$ be the semigroup generated by a set of integers $\{d_1, \ldots, d_m\}$ such that

$$1 < d_1 < \ldots < d_m\ , \quad \gcd(d_1, \ldots, d_m) = 1\ . \tag{1}$$

For short we denote the tuple $(d_1, \ldots, d_m)$ by $\mathbf{d}^m$ and consider the generating function $\Phi\,(\mathbf{d}^m; z)$

$$\Phi\,(\mathbf{d}^m; z) = \sum_{s_i\,\in\,\Delta(\mathbf{d}^m)} z^{s_i} = z + z^{s_2} + \ldots + z^{s_{G(\mathbf{d}^m)}}\ , \tag{2}$$

for the set $\Delta\,(\mathbf{d}^m)$ of the integers s which are unrepresentable as $s = \sum_{i=1}^m x_i d_i,\ x_i \in \mathbb{N} \cup \{0\}$

$$\Delta\,(\mathbf{d}^m) = \{s_1, s_2, \ldots, s_{G(\mathbf{d}^m)}\}\ , \quad s_1 = 1\ . \tag{3}$$

The integer $G\,(\mathbf{d}^m)$ is known as the genus for semigroup $\mathsf{S}\,(\mathbf{d}^m)$. Recall the relation of $\Phi\,(\mathbf{d}^m; z)$ with the Hilbert series $H(\mathbf{d}^m; z)$ of a graded monomial subring $\mathsf{k}\left[z^{d_1}, \ldots, z^{d_m}\right]$ [1]

$$H(\mathbf{d}^m; z) + \Phi(\mathbf{d}^m; z) = \frac{1}{1-z}\ , \quad \text{where} \quad H(\mathbf{d}^m; z) = \sum_{s\,\in\,\mathsf{S}(\mathbf{d}^m)} z^s = \frac{Q(\mathbf{d}^m; z)}{\prod_{j=1}^m (1 - z^{d_j})}\ , \tag{4}$$

and $Q(\mathbf{d}^m;z)$ is a polynomial in z. The calculation of the power sums

$$g_n\left(\mathbf{d}^m\right)=\sum_{s_i\,\in\,\Delta(\mathbf{d}^m)} s_i^n \tag{5}$$

was performed in [2] for $m=2$

$$g_n\left(\mathbf{d}^2\right)=\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n+1}\sum_{l=0}^{n+1-k}\binom{n+2}{k}\binom{n+2-k}{l}B_kB_ld_1^{n+1-k}d_2^{n+1-l}-\frac{B_{n+1}}{n+1}\;, \tag{6}$$

where B_k are the Bernoulli numbers. The formula (6) generalizes the known Sylvester's expression [3] for $G\left(\mathbf{d}^2\right)=g_0\left(\mathbf{d}^2\right)$ and further results for $n=1$ [4] and $n=2,3$ [5]

$$\begin{aligned}
g_0\left(\mathbf{d}^2\right) &= \frac{1}{2}(d_1-1)(d_2-1)\;,\\
g_1\left(\mathbf{d}^2\right) &= \frac{g_0\left(\mathbf{d}^2\right)}{6}\,(2d_1d_2-d_1-d_2-1)\;,\\
g_2\left(\mathbf{d}^2\right) &= \frac{g_0\left(\mathbf{d}^2\right)}{6}\,d_1d_2(d_1d_2-d_1-d_2)\;,\\
g_3\left(\mathbf{d}^2\right) &= \frac{g_0\left(\mathbf{d}^2\right)}{60}\,\left[(1+d_1)(1+d_2)\left(1+d_1^2+d_2^2+6d_1^2d_2^2\right)-15d_1d_2(d_1+d_2)\right]\;.
\end{aligned}$$

As for higher dimensions, $m\geq 3$, the first two sums, $g_0\left(\mathbf{d}^3\right)$ and $g_1\left(\mathbf{d}^3\right)$, were found in [5]. The use was made of the explicit expression for the Hilbert series $H(\mathbf{d}^3;z)$ of a graded subring for semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ which was recently established [5].

In this paper we derive the formula for the power sum $g_n\left(\mathbf{d}^3\right)$ related to the symmetric and non-symmetric semigroups $S\left(\mathbf{d}^3\right)$. This will be done by applying to the relation (4) an approach based on the generating function of the Bernoulli polynomials of higher order.

Following Johnson [6] recall the definition of *the minimal relations* for given $\mathbf{d}^3=(d_1,d_2,d_3)$

$$a_{11}d_1=a_{12}d_2+a_{13}d_3\;,\quad a_{22}d_2=a_{21}d_1+a_{23}d_3\;,\quad a_{33}d_3=a_{31}d_1+a_{32}d_2\;,\quad \text{where} \tag{7}$$

$$\begin{aligned}
a_{11}&=\min\left\{v_{11}\mid v_{11}\geq 2,\ v_{11}d_1=v_{12}d_2+v_{13}d_3,\ v_{12},v_{13}\in\mathbb{N}\cup\{0\}\right\}\;,\\
a_{22}&=\min\left\{v_{22}\mid v_{22}\geq 2,\ v_{22}d_2=v_{21}d_1+v_{23}d_3,\ v_{21},v_{23}\in\mathbb{N}\cup\{0\}\right\}\;,\\
a_{33}&=\min\left\{v_{33}\mid v_{33}\geq 2,\ v_{33}d_3=v_{31}d_1+v_{32}d_2,\ v_{31},v_{32}\in\mathbb{N}\cup\{0\}\right\}\;.
\end{aligned} \tag{8}$$

The auxiliary invariants a_{ij} are uniquely defined by (8) and

$$\gcd(a_{11},a_{12},a_{13})=1\;,\quad \gcd(a_{21},a_{22},a_{23})=1\;,\quad \gcd(a_{31},a_{32},a_{33})=1\;. \tag{9}$$

According to [5] the numerator of the Hilbert series for non-symmetric semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ reads

$$Q(\mathbf{d}^3;z) = 1-\sum_{i=1}^{3} z^{a_{ii}d_i} + z^{1/2\left[\langle\mathbf{a},\mathbf{d}\rangle - J\left(\mathbf{d}^3\right)\right]} + z^{1/2\left[\langle\mathbf{a},\mathbf{d}\rangle + J\left(\mathbf{d}^3\right)\right]}\ , \quad \text{where} \tag{10}$$

$$J^2\left(\mathbf{d}^3\right) = \langle\mathbf{a},\mathbf{d}\rangle^2 - 4\sum_{i>j} a_{ii}a_{jj}d_id_j + 4d_1d_2d_3\ , \quad \langle\mathbf{a},\mathbf{d}\rangle = \sum_{i=1}^{3} a_{ii}d_i\ . \tag{11}$$

The case of symmetric semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ is much simpler. Here two off-diagonal elements in one column of the matrix a_{ij} vanish, e.g. $a_{13}=a_{23}=0$ that results in $a_{11}d_1=a_{22}d_2$. The numerator of the Hilbert series for symmetric semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ with above symmetry is given by

$$Q_s(\mathbf{d}^3;z) = (1-z^{a_{22}d_2})(1-z^{a_{33}d_3})\ . \tag{12}$$

Move on to calculation of the power sums $g_n\left(\mathbf{d}^3\right)$. First, denote $z=e^t$ and represent (4) as follows

$$\sum_{s_i\,\in\,\Delta(\mathbf{d}^3)} e^{s_it} = \frac{1}{1-e^t} - \frac{Q(\mathbf{d}^3;e^t)}{\left(1-e^{d_1t}\right)\left(1-e^{d_2t}\right)\left(1-e^{d_3t}\right)}\ , \tag{13}$$

and apply the sequence of identities

$$\sum_{s_i\,\in\,\Delta(\mathbf{d}^3)} e^{s_it} = \sum_{s_i\,\in\,\Delta(\mathbf{d}^3)} \sum_{k=0}^{\infty} \frac{s_i^n t^n}{n!} = \sum_{k=0}^{\infty} \frac{t^n}{n!} \sum_{s_i\,\in\,\Delta(\mathbf{d}^3)} s_i^n = \sum_{n=0}^{\infty} g_n\left(\mathbf{d}^3\right)\frac{t^n}{n!}\ . \tag{14}$$

Thus, $g_n\left(\mathbf{d}^3\right)$ could be found by expanding the right hand side of (13) in the power series.

To present the result in compact form we use a definition [7] of the Bernoulli polynomials of higher order

$$\frac{e^{xt}}{\prod_{i=1}^m (e^{d_it}-1)} = \frac{1}{\pi_d}\sum_{n=0}^{\infty}\frac{t^{n-m}}{n!}B_n^{(m)}(x|\mathbf{d}^m)\ , \quad \pi_d = \prod_{i=1}^m d_i, \tag{15}$$

satisfying the recursion relation

$$B_n^{(m)}(x|\mathbf{d}^m) = \sum_{p=0}^{n}\binom{n}{p} d_m^p B_p B_{n-p}^{(m-1)}(x|\mathbf{d}^{m-1}),\quad B_n^{(1)}(x|d) = d^n B_n\left(\frac{x}{d}\right), \tag{16}$$

where $B_n(x)$ denotes the regular Bernoulli polynomial. Because each term in the right hand side of (13) has the form of the left hand side of (15), it is easy to write the answer as sum of the Bernoulli polynomials of higher order.

First, contribution of the term $1/(1-z)$ to $g_n\left(\mathbf{d}^3\right)$ is found trivially

$$\frac{1}{1-e^t} = -\sum_{n=0}^{\infty}\frac{t^{n-1}}{n!}B_n^{(1)}(0|1) = -\sum_{n=0}^{\infty}\frac{t^{n-1}}{(n-1)!}\frac{B_n}{n}, \tag{17}$$

so that the corresponding term in $g_n(\mathbf{d}^3)$ is $-B_{n+1}/(n+1)$. Consider a general term for $n=3$

$$\frac{e^{tx}}{(1-e^{d_1t})(1-e^{d_2t})(1-e^{d_3t})} = -\frac{1}{d_1d_2d_3}\sum_{n=0}^{\infty}\frac{t^{n-3}}{n!}B_n^{(3)}(x|\mathbf{d}^3), \tag{18}$$

thus its contribution to $g_n(\mathbf{d}^3)$ reads:

$$g(x;\mathbf{d}^3) = -\frac{n!}{(n+3)!\ d_1d_2d_3}B_{n+3}^{(3)}(x|\mathbf{d}^3). \tag{19}$$

The Bernoulli polynomial of higher order $B_{n+3}^{(3)}(x|\mathbf{d}^3)$ can be expanded into the triple sum over the Bernoulli numbers

$$B_{n+3}^{(3)}(x|\mathbf{d}^3) = \sum_{j=0}^{n+3}\sum_{k=0}^{j}\sum_{l=0}^{k}\binom{n+3}{j}\binom{j}{k}\binom{k}{l}d_1^{k-l}d_2^{j-k}d_3^{n+3-j}B_{k-l}B_{j-k}B_{n+3-j}x^l, \tag{20}$$

which for $x=0$ reduces to the double sum

$$B_{n+3}^{(3)}(0|\mathbf{d}^3) = \sum_{j=0}^{n+3}\sum_{k=0}^{j}\binom{n+3}{j}\binom{j}{k}d_1^{k}d_2^{j-k}d_3^{n+3-j}B_kB_{j-k}B_{n+3-j}. \tag{21}$$

The expression for $g_n(\mathbf{d}^3)$ for the non-symmetric semigroups has the form

$$g_n^{(n)}(\mathbf{d}^3) = -\frac{B_{n+1}}{n+1} + \frac{n!}{(n+3)!\ d_1d_2d_3}\Big[B_{n+3}^{(3)}(0|\mathbf{d}^3) - B_{n+3}^{(3)}(a_{11}d_1|\mathbf{d}^3) - B_{n+3}^{(3)}(a_{22}d_2|\mathbf{d}^3) - \tag{22}$$
$$B_{n+3}^{(3)}(a_{33}d_3|\mathbf{d}^3) + B_{n+3}^{(3)}(1/2\left[\langle\mathbf{a},\mathbf{d}\rangle - J\left(\mathbf{d}^3\right)\right]|\mathbf{d}^3) + B_{n+3}^{(3)}(1/2\left[\langle\mathbf{a},\mathbf{d}\rangle + J\left(\mathbf{d}^3\right)\right]|\mathbf{d}^3)\Big].$$

In case of the symmetric semigroups we obtain

$$g_n^{(s)}(\mathbf{d}^3) = -\frac{B_{n+1}}{n+1} + \frac{n!}{(n+3)!\ d_1d_2d_3}\Big[B_{n+3}^{(3)}(0|\mathbf{d}^3) - B_{n+3}^{(3)}(a_{22}d_2|\mathbf{d}^3) - B_{n+3}^{(3)}(a_{33}d_3|\mathbf{d}^3) +$$
$$B_{n+3}^{(3)}(a_{22}d_2 + a_{33}d_3|\mathbf{d}^3)\Big]. \tag{23}$$

Thus, for the symmetric semigroup $\mathsf{S}(\mathbf{d}^3)$ the first three sums read

$$\begin{aligned}
2g_0^{(s)}(\mathbf{d}^3) &= 1 - s_d + \widetilde{\langle\mathbf{a},\mathbf{d}\rangle}\ , \quad s_d = \sum_{i=1}^{3} d_i\ , \quad \widetilde{\langle\mathbf{a},\mathbf{d}\rangle} = \langle\mathbf{a},\mathbf{d}\rangle - a_{11}d_1\ , && (24)\\
12g_1^{(s)}(\mathbf{d}^3) &= \left(s_d - \widetilde{\langle\mathbf{a},\mathbf{d}\rangle}\right)\left(s_d - 2\widetilde{\langle\mathbf{a},\mathbf{d}\rangle}\right) + (d_1d_2 + d_1d_3 + d_2d_3) - \pi_d - 1\ , && (25)\\
12g_2^{(s)}(\mathbf{d}^3) &= (s_d - \widetilde{\langle\mathbf{a},\mathbf{d}\rangle})\left(s_d\widetilde{\langle\mathbf{a},\mathbf{d}\rangle} - \widetilde{\langle\mathbf{a},\mathbf{d}\rangle}^2 - (d_1d_2 + d_1d_3 + d_2d_3) + \pi_d\right)\ . && (26)
\end{aligned}$$

In the non-symmetric case we obtain:

$$
\begin{aligned}
2g_0^{(n)}(\mathbf{d}^3) &= 1 - s_d - \pi_a + \langle \mathbf{a}, \mathbf{d} \rangle \ , \quad \pi_a = a_{11}a_{22}a_{33} \ , && (27)\\
12g_1^{(n)}(\mathbf{d}^3) &= \langle \mathbf{a}, \mathbf{d} \rangle (2\langle \mathbf{a}, \mathbf{d} \rangle - 3s_d - 2\pi_a) + s_d(s_d + 3\pi_a) - \sum_{i \neq j}^{3} a_{ii}a_{jj}d_i d_j + \\
&\quad (d_1 d_2 + d_1 d_3 + d_2 d_3) + \pi_d - 1 \ , && (28)\\
12g_2^{(n)}(\mathbf{d}^3) &= \sum_{i=1}^{3} A_i \left[(2A_i + 1)d_i^3 - \frac{\pi_a}{2}(A_i + 2)d_i^2 + \pi_d d_i + \frac{\pi_d}{2}(2A_i + 1) \right] + \\
&\quad \sum_{i,j=1}^{3} \left[C_{ij} d_i^2 d_j - \frac{\pi_a}{2} B_{ij} d_i d_j - \frac{\pi_d}{2} F_{ij} \right] \ , && (29)
\end{aligned}
$$

where

$$
A_i = a_{ii} - 1 \ , \quad B_{ij} = A_i A_j - A_i - A_j \ , \quad C_{ij} = A_i(A_i A_j - A_i - 1) \ , \quad F_{ij} = A_i(2A_j + 1) \ .
$$

We finish the paper with a compact version of (6) through the Bernoulli polynomials of higher orders

$$
g_n(\mathbf{d}^2) = -\frac{B_{n+1}}{n+1} - \frac{1}{d_1 d_2 (n+1)(n+2)} [B_{n+2}^{(2)}(0|\mathbf{d}^2) - B_{n+2}^{(2)}(d_1 d_2|\mathbf{d}^2)] \ , \tag{30}
$$

where

$$
B_{n+2}^{(2)}(x|\mathbf{d}^2) = \sum_{j=0}^{n+2} \sum_{k=0}^{j} \binom{n+2}{j} \binom{j}{k} d_1^{j-k} d_2^{n+2-j} B_{j-k} B_{n+2-j} x^k \ . \tag{31}
$$

References

[1] R. P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser Boston, 2nd ed., (1996)

[2] Ö. J. Rödseth, *A note on Brown and Shiue's paper on a Remark Related to the Frobenius Problem*, Fibonacci Quarterly, **32**, 407 (1994)

[3] J. J. Sylvester, *Mathematical Questions with Their Solutions*, Educational Times, **41**, 171 (1884)

[4] T. C. Brown and P. J. Shiue, *A Remark Related to the Frobenius Problem*, Fibonacci Quarterly, **31**, 32 (1993)

[5] L. G. Fel, *Frobenius Problem for Semigroups* $S\,(d_1, d_2, d_3)$,
preprint, (2004), [`http://arXiv.org/abs/math.NT/0409331`]

[6] S. M. Johnson, *A Linear Diophantine Problem*,
Canad. J. Math., **12**, 390 (1960)

[7] H. Bateman and A. Erdelýi, *Higher Transcendental Functions*, Vol.1, Ch.1,
McGraw-Hill Book Co., NY (1953)