Stability of Localized Solutions in a Subcritically Unstable Pattern-Forming System under a Global Delayed Control

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Abstract

The formation of spatially localized patterns in a system with subcritical instability under feedback control with delay is investigated within the framework of globally-controlled Ginzburg-Landau equation. It is shown that feedback control can stabilize spatially-localized solutions. With the increase of delay, these solutions undergo oscillatory instability that, for large enough control strength, result in the formation of localized oscillating pulses. With further increase of delay the pulse amplitude blows up due to merging of stable and unstable limit cycles.

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I. INTRODUCTION

Various nonlinear extended systems are subject to saturable monotonic short-wave ("Turing") instabilities leading to the formation of stationary patterns[1–3]. The most well know examples are Rayleigh-Bénard convection and Turing patterns in reaction-diffusion systems. In the one-dimensional case, the generic equation governing the pattern envelope (amplitude) function $A(x,t)$ is the Ginzburg-Landau equation,

$$\frac{\partial A}{\partial t} = \mu A + D\frac{\partial^2 A}{\partial x^2} - \kappa |A|^2 A, \tag{1}$$

with real coefficients μ, $D > 0$ and $\kappa > 0$. In the subcritical region, $\mu < 0$, the solution $A = 0$ corresponding to the equilibrium state is stable. In the supercritical region, the time evolution of the amplitude function is characterized by a monotonic decrease of a Lyapunov functional ("Ginzburg-Landau free energy") and asymptotically leads to a stable stationary spatially-periodic solution with a wavenumber within the Eckhaus stability interval, $k^2 < \mu/(3D)$[4]. Other kinds of stationary solutions, such as spatially-localized and quasiperiodic solutions, are unstable. This provides the explanation of the formation of ordered patterns from disordered initial conditions.

Some nonlinear systems exhibit non-saturable (subcritical) instabilities, corresponding to the case $\kappa < 0$ in (1). In this case, the Lyapunov functional is not bounded from below, and the solution of (1) blows up in a finite time. Of course, the description of the underlying physical problem by means of the weakly nonlinear equation (1) fails in this case. However, the blow-up of solutions can be prevented by a nonlinear feedback control. An example of such a control, which leads to the following modification of the Ginzburg-Landau equation,

$$\frac{\partial A}{\partial t} = (\mu - K[A])A + D\frac{\partial^2 A}{\partial x^2} - \kappa |A|^2 A, \tag{2}$$

where $K[A] = p \max_x |A(x,t)|$, $p > 0$, has been considered in[5], where it was applied for modeling the suppression of a morphological instability of a solidification front. It was shown that the stability properties of stationary solutions of (2) significantly differed from those of (1). All the spatially-periodic solutions of (1) turned out to be unstable, while the only stable solution observed in numerical simulations corresponded to a localized one.

Usually, in systems with feedback control, there is a delay between the measurement of the system parameters by sensors and the application of control action by actuators. In some systems this delay is small and can be neglected. The analysis described above is valid for this case. In the present paper, we consider the general case when a delay in feedback control is present. Thus, we consider a more general nonlinear control, $K[A] = p \max_x |A(x, t - \tau)|$, $p > 0$ where $\tau =$ const is the control delay. Obviously, the stationary solutions of (2) are not affected by the delay. However, the control delay may change the stability properties of solutions and create new dynamic regimes.

The paper is organized as follows. In Sec. 2, we present stationary localized solutions for Eqs. (1) and (2). In Sec. 3, we perform the linear stability analysis of these solutions. We will show that the localized solutions are unstable in the absence of control, while in the presence of an undelayed control there exist two branches of solutions, one of which is always stable and another one is unstable. We will also show that the delay of control may lead to an oscillatory instability of the localized solutions, and find the linear stability boundary $\tau(p)$. Sec. 4 is devoted to nonlinear simulations of finite-amplitude pulse oscillations. Sec. 5 contains concluding remarks.

II. STATIONARY LOCALIZED SOLUTIONS

For $\kappa < 0$, upon rescaling, one can rewrite Eq. (2) as

$$\frac{\partial A}{\partial t} = (s - K[A])A + \frac{\partial^2 A}{\partial x^2} + |A|^2 A, \tag{3}$$

where

$$s = \text{sign}(\mu), \;\; K[A] = p \max_x |A(t - \tau)|,$$

without a loss of generality.

Eq.(3) has a stationary localized solution,

$$A(x) = A_0(x) = R(x - x_0)e^{i\Theta}, \tag{4}$$

where x_0 and Θ are arbitrary constants,

$$R(y) = \sqrt{2}k(q)\,\text{sech}[k(q)y], \;\; -\infty < y < \infty, \tag{5}$$

and $k(q)$ is a positive root of the quadratic equation,

$$k^2 - 2kq + s = 0, \;\; q = p/\sqrt{2}. \tag{6}$$

In the subcritical region, $s = -1$, there exists only one solution branch,

$$k(q) = q + \sqrt{q^2 + 1}, \ -\infty < q < \infty. \tag{7}$$

Specifically, for $q = 0$, i.e. in the absence of control, the localized solution has the form

$$R(y) = \sqrt{2}\,\mathrm{sech}\, y, \ -\infty < y < \infty. \tag{8}$$

In the supercritical region, $s = 1$, there are two branches of solutions,

$$k(q) = q \pm \sqrt{q^2 - 1}, \ q \geq 1. \tag{9}$$

Note that for any localized solution the effective linear growth rate,

$$\sigma_0 = s - K[A_0] = s - 2qk(q) = -[k(q)]^2 < 0, \tag{10}$$

in the whole region of the localized solution existence.

III. STABILITY OF LOCALIZED SOLUTIONS

A. Formulation of the problem

Obviously, the stability of a localized solution does not depend on x_0 and Θ. Below, we set $x_0 = \Theta = 0$, and consider real $A_0(x) = R(x)$. In order to investigate the stability, we consider the evolution of a disturbance on the background of the stationary solution. Linearizing Eq. (3) around the localized solution (4),

$$A(x, t) = A_0(x) + \tilde{A}(x, t), \tag{11}$$

we find:

$$\begin{aligned} \frac{\partial \tilde{A}(x,t)}{\partial t} &= \frac{\partial^2 \tilde{A}(x,t)}{\partial x^2} + [s - 2qk(q) + 2A_0^2(x)]\tilde{A}(x,t) \\ &\quad + A_0^2(x)\tilde{A}^*(x,t) - 2qk(q)\mathrm{Re}\tilde{A}(0, t - \tau). \end{aligned} \tag{12}$$

It is assumed that $|\tilde{A}(x,t)|$ is bounded for $x \to \pm\infty$. Define $\tilde{A}(x,t) = \tilde{A}_r(x,t) + i\tilde{A}_i(x,t)$, where $\tilde{A}_r$ and $\tilde{A}_i$ are real functions. The problems for $\tilde{A}_r$ and $\tilde{A}_i$ are decoupled:

$$\begin{aligned} \frac{\partial \tilde{A}_r(x,t)}{\partial t} &= \frac{\partial^2 \tilde{A}_r(x,t)}{\partial x^2} + [s - 2qk(q) + 3A_0^2(x)]\tilde{A}_r(x,t) \\ &\quad - 2qk(q)\tilde{A}_r(0, t - \tau); \end{aligned} \tag{13}$$

$$\frac{\partial \tilde{A}_i(x,t)}{\partial t} = \frac{\partial^2 \tilde{A}_i(x,t)}{\partial x^2} + [s - 2qk(q) + A_0^2(x)]\tilde{A}_i(x,t). \tag{14}$$

Here,

$$A_0(x) = \sqrt{2}k(q)\,\mathrm{sech}\,[k(q)x].$$

Introduce a new coordinate, $z \equiv k(q)x$, and consider normal modes,

$$\tilde{A}_r(x,t) = u(z)e^{\sigma t},\ \tilde{A}_i(x,t) = v(z)e^{\sigma t}.$$

Taking into account relation (6), one obtains an equation which is valid in both subcritical and supercritical cases:

$$k^2u'' + \left(-k^2 - \sigma + \frac{6k^2}{\cosh^2 z}\right)u = 2kq\frac{u(0)e^{-\sigma\tau}}{\cosh z};\ |u| < \infty,\ z \to \pm\infty; \tag{15}$$

$$k^2v'' + \left(-k^2 - \sigma + \frac{2k^2}{\cosh^2 z}\right)v = 0;\ |v| < \infty,\ z \to \pm\infty, \tag{16}$$

where $'$ means the differentiation with respect to z.

The problem (15) describes amplitude disturbances of the localized solution, while (16) describes its phase disturbances.

B. Phase disturbances

Let us start with the problem (16). Rewrite it as

$$-v'' + (1 - 2\cosh^{-2} z)v = -(\sigma/k^2)v;\ |v| < \infty,\ z \to \pm\infty, \tag{17}$$

to obtain the well-known eigenvalue problem for the Schrödinger equation, which is exactly solvable (see, e.g.,[6] or[7]). The continuum spectrum of the problem is $-(\sigma/k^2) > 1$, hence, it does not produce any instability. The only discrete eigenvalue is $\sigma = 0$, with the eigenfunction

$$v(z) = \mathrm{sech}\, z,$$

which corresponds to an infinitesimal change of Θ in (4).

C. Amplitude disturbances

In the present subsection, we analyze the non-local eigenvalue problem (15).

1. Stability in the absence of the control

In the case $q = 0$ (no control) , the localized solution exists only in the subcritical region, $s = -1$, and is described by Eq. (4) with $k = 1$. The eigenvalue problem (15) can be written as

$$-u'' + (1 - 6\,\mathrm{sech}^2\, z)u = -\sigma u;\ |u| < \infty,\ z \to \pm\infty. \tag{18}$$

Again, the continuum spectrum of the problem is located at $\sigma < -1$ and does not produce any instability. The discrete spectrum includes two eigenvalues[6],[7]:

$$\sigma = 0,\ u = \sinh z\, \mathrm{sech}^2\, z;$$

$$\sigma = 3,\ u = \mathrm{sech}^2\, z.$$

The first mode corresponds to a translation of the localized solution (an infinitesimal change of x_0 in (4)). The second mode results in the instability of the subcritical localized solution in the absence of control.

2. Stability in the presence of the control

Now consider Eq. (15) in the case when control is present, $q \neq 0$. Obviously, $\sigma < -k^2 < 0$ for any disturbances which do not decay as $z \to \infty$. Hence, for the stability analysis it is sufficient to consider localized solutions with $\mathrm{Re}\,\sigma > -k^2$. Due to the symmetry of Eq. (15), any eigenfunctions can be represented by either even or odd functions of z. For odd eigenfunctions, $u(0) = 0$, and one returns to the uncontrolled case discussed above. Therefore, later on we consider only even solutions of Eq. (15).

a. Analytical solution of the linear stability problem One can fix the norm of the eigenfunction $u(z)$ by the condition $u(0) = 1$, and present the eigenvalue problem in the form

$$u'' + \left(\frac{6}{\cosh^2 z} - r^2\right) u = m\frac{\exp[(1 - r^2)\tau/(m - 1)]}{\cosh z},\ -\infty < z < \infty; \tag{19}$$

$$|u| \to 0,\ z \to \pm\infty; \tag{20}$$

where

$$r^2 = \frac{\sigma + k^2}{k^2},\ m = \frac{s + k^2}{k^2}.$$

According to (7), in the subcritical region , $s = -1$,

$$m = \frac{2q}{q + \sqrt{q^2+1}}.$$

Hence, $0 < m < 1$ for the stabilizing control ($q > 0$), and $-\infty < m < 0$ for the destabilizing control ($q < 0$). In the supercritical region, $s = 1$,

$$m = \frac{2q}{q \pm \sqrt{q^2-1}},$$

where $q \geq 1$ (see (9)). One can see that $1 < m \leq 2$ for the upper branch and $2 \leq m < \infty$ for the lower branch. Thus, the stability of localized solutions in all the cases mentioned in Sec. 2 can be studied by means of Eq. (19).

The eigenvalue σ is above the continuous spectrum if $Re(r^2) > 0$. The instability corresponds to $Re(r^2) > 1$.

The general solution of Eq. (19) can be written as

$$u(z) = u_0(z) + u_p(z),$$

where u_0 is the general solution of the homogeneous equation and u_p is a particular solution of the inhomogeneous equation. Since we are interested only in even solutions of the problem, it is sufficient to consider the region $0 \leq z < \infty$.

The general solution of the homogeneous equation is

$$u_0(z) = C_+^h \; P_2^r(\tanh z) + C_-^h \; P_2^{-r}(\tanh z), \tag{21}$$

where $P_n^m(x)$ denotes the associated Legendre polynomial. The particular solution of the inhomogeneous equation can be found using variation of parameters, which gives

$$u_0(z) = C_+^i(z) \; P_2^r(\tanh z) + C_-^i(z) \; P_2^{-r}(\tanh z), \tag{22}$$

where

$$\begin{aligned} C_\pm^i(z) = \pm\frac{\pi m \exp[(1-r^2)\tau/(m-1)]}{2\sin \pi r} \int \frac{P_2^{\mp r}(\tanh z)}{\cosh z} dz = \\ \pm\frac{\pi m \exp[(1-r^2)\tau/(m-1)]}{2\sin \pi r} \int \frac{P_2^{\mp r}(y)}{\sqrt{1-y^2}} dy \; . \end{aligned} \tag{23}$$

The computation of the integral in (23) gives

$$\begin{aligned} C_\pm^i \;&=\; \pm\frac{\pi m \exp[(1-r^2)\tau]/(m-1)}{2\Gamma(3 \pm r)\sin \pi r} \\ &\times\; \Big[3\left(I_3^{(\mp r+3)/2}(w) - 2I_3^{(\mp r+1)/2}(w) + I_3^{(\mp r-1)/2}(w)\right) \\ &\pm\; 3p\left(I_2^{(\mp r+1)/2}(w) - I_2^{(\mp r-1)/2}(w)\right) + (r^2-1)I_1^{(\mp r-1)/2}(w)\Big] \; , \end{aligned} \tag{24}$$

where $w = (1 + y)/(1 - y) = (1 + \tanh z)/(1 - \tanh z)$, and

$$I_n^\alpha(w) = \int \frac{w^\alpha}{(w+1)^n} dw = \frac{w^{1+\alpha}}{1+\alpha} \, {}_2F_1(n, 1+\alpha; 2+\alpha; -w) \ . \tag{25}$$

Here ${}_2F_1$ is a confluent hypergeometric function. Thus, the general solution of the problem reads

$$u(z) = (C_+^h + C_+^i) P_2^r(\tanh z) + (C_-^h + C_-^i) P_2^{-r}(\tanh z). \tag{26}$$

Condition (20) leads to following values of the coefficients:

$$C_+^h = -\frac{\pi m \exp[(1-r^2)\tau/(m-1)]}{2\Gamma(3+r)\sin \pi r} \frac{\pi(1-r^2)}{2} \sec \frac{\pi r}{2}, \ C_-^h = 0. \tag{27}$$

Finally, applying the condition $u(0) = 1$ to Eqs. (24), (26) and (27), and using the properties of the Γ-function and hypergeometric functions[8], one arrives at the following relation,

$$\frac{m \exp[(1-r^2)\tau/(m-1)]}{2r(4-r^2)} \left\{ 3r + \frac{r^2-1}{2} \left[\psi\left(\frac{r+1}{4}\right) - \psi\left(\frac{r+3}{4}\right) \right] \right\} = 1, \tag{28}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of Γ-function. Eq. (28) describes the dependence of the growth rate on the control parameter for both subcritical and supercritical regions in an implicit form.

b. Stability of localized solutions under control without delay. First, let us consider the case of undelayed control, $\tau = 0$. In this case,

$$\frac{m}{2r(4-r^2)} \left\{ 3r + \frac{r^2-1}{2} \left[\psi\left(\frac{r+1}{4}\right) - \psi\left(\frac{r+3}{4}\right) \right] \right\} = 1. \tag{29}$$

For monotonic disturbances (real r), the dependence $m(r)$ can be found explicitly; it is shown in Figs. 1 and 2. One can see that for any $m < 2$ there are two values of r, one of which is larger than 1. This means that the localized solutions in the subcritical region, as well as the upper branch of the localized solutions in the supercritical region are unstable.

In the region $2 < m < m_* = 2.02193$, both values of r are real and less than 1. For $m > m_*$ the two eigenvalues r are complex conjugate, with $Re(r^2) < 1$ (see Fig. 2). For large m, the leading order terms of the asymptotic expansion for $r(m)$ can be written in the form

$$r = \frac{\pi(m-4)}{4} \, e^{-\pi\sqrt{m-5}/2} + i\sqrt{m-5}. \tag{30}$$

Hence, the stability condition $Re(r^2) < 1$ is not violated in this case.

Therefore, the lower branch of the localized solutions in the supercritical region is stable in the whole region of its existence, $m > 2$, i.e. for $p > \sqrt{2}$. The stability of the localized

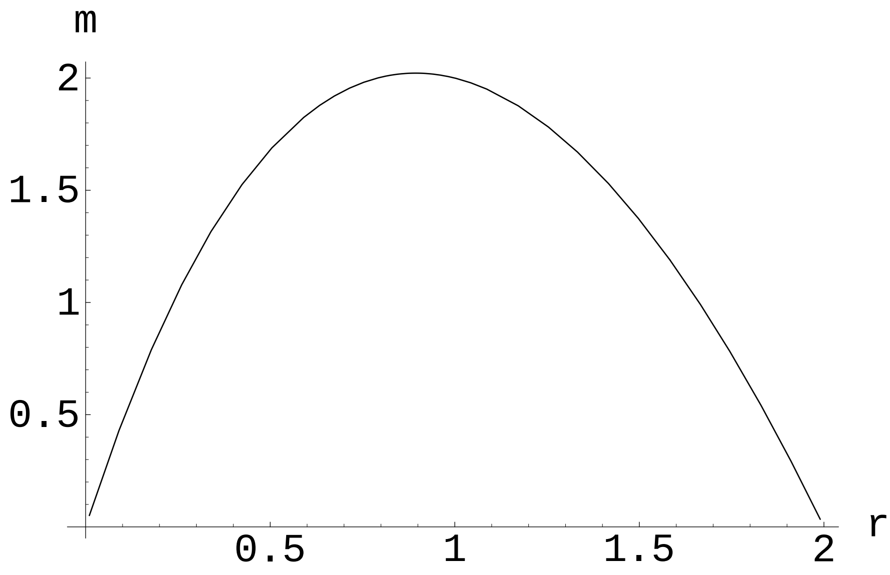

FIG. 1: Dependence $m(r)$ defined by (29).

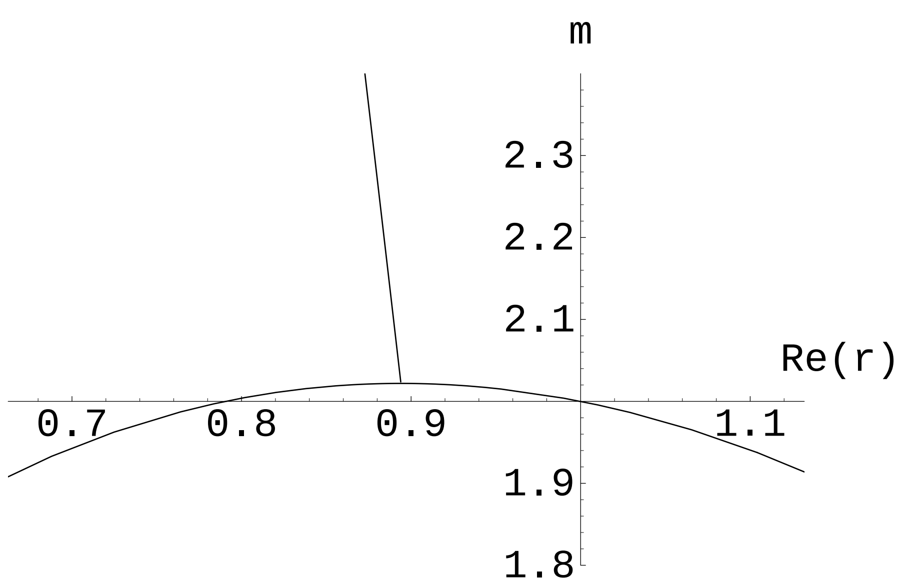

FIG. 2: Real parts of $r(m)$ defined by (29) for $m < m_*$ (two real eigenvalues) and $m > m_*$ (two complex conjugate eigenvalues).

solutions under the global control was recently observed in numerical simulations[5]. Note that the neutral disturbance ($r = 1$) corresponding to the merging point of the two branches ($m = 2$) can be expressed by means of elementary functions, $u(z) = \operatorname{sech} z - z \sinh z \operatorname{sech}^2 z$.

Typical eigenfunctions $u(z)$ corresponding to the localized disturbances at the lower (stable) branch ($r = 0.96$) and the upper (unstable) branch ($r = 1.02$) are shown in Fig. 3.

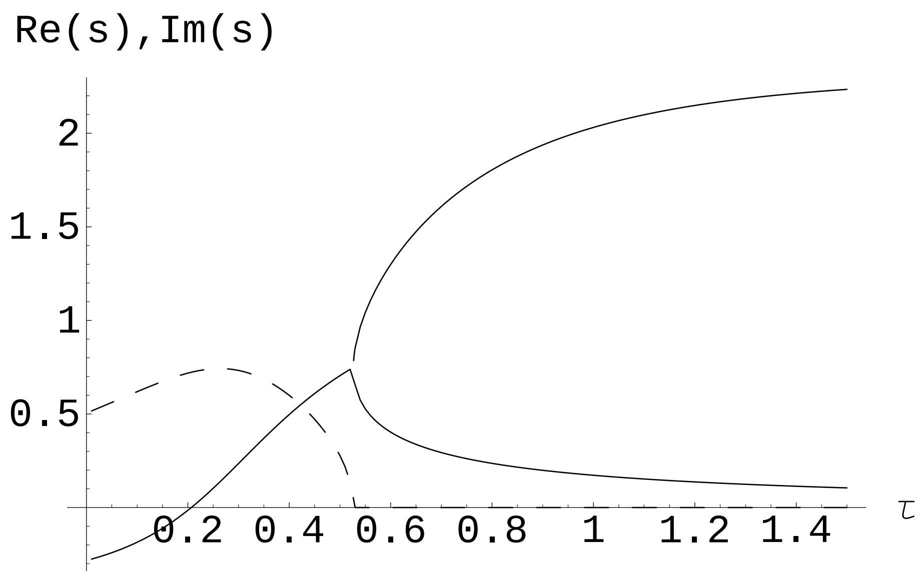

FIG. 3: Typical dispersion curves defined by (28), showing the real part (solid lines) and imaginary part (dashed line) of the perturbation growth rate σ as functions of the control delay τ.

Note that the eigenfunctions are even, as expected.

c. Stability of localized solutions under delayed control Let us consider now the case of delayed control, $\tau \neq 0$, described by (28). Obviously, the monotonic stability boundary, $r^2 = 1$, is not changed by the delay. Hence, the boundary between monotonically stable and monotonically unstable solutions, $m = 2$, is unchanged. Therefore, the localized solutions at the upper branch are unstable. However, the delay can produce an oscillatory instability of the supercritical localized solution corresponding to the lower branch. A typical dependence of the growth rate σ on the delay parameter τ for a fixed value of p is shown in Fig. 3. The growth rate of a monotonic mode cannot cross the value $\sigma = 0$ at a finite value of τ, but the real part of the growth rate of the oscillatory mode does cross zero. Note that for sufficiently large τ the pair of complex conjugate eigenvalues with $Re(\sigma) > 0$ is transformed into a pair of real positive eigenvalues, i.e. the instability of the stationary localized solution becomes monotonic.

The region of the localized solution stability is presented in Fig. 4. The end point of the oscillatory instability boundary corresponds to $p = \sqrt{2}$ and $\tau = (1 - \ln 2)/3 \approx 0.1$. For $p \gg 1$, $\tau \to \pi/2$.

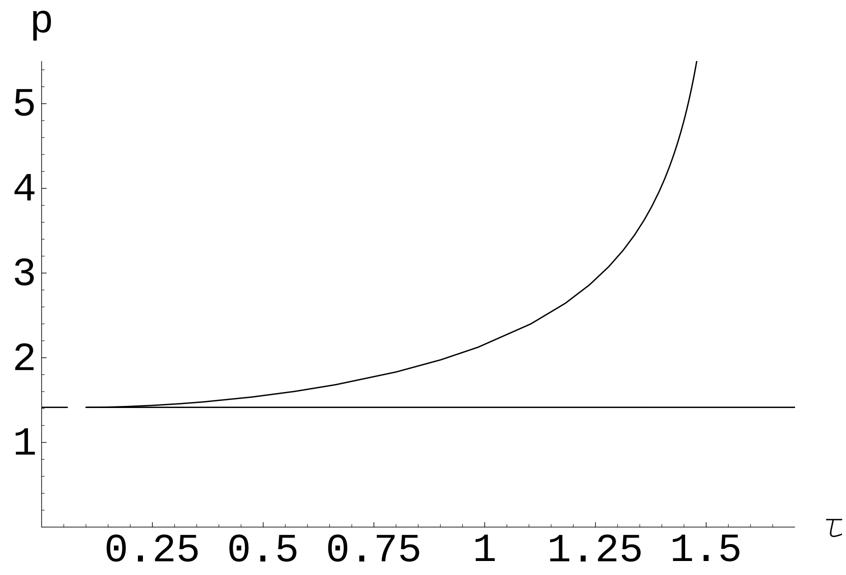

FIG. 4: Stability boundaries of the localized solution (4)-(6) in (τ, p)-plane.

IV. NUMERICAL SIMULATIONS

We have performed numerical simulations of Eq. (3) with delayed control term, by means of a pseudo-spectral code with periodic boundary conditions and time integration in Fourier space, using Crank-Nicholson scheme the linear operator and Adams-Bashforth one for the nonlinear one. Fig. 5 shows the spatio-temporal diagrams of solutions for the parameter values inside the stability domain and outside it (see Fig.4). For parameters inside the stability domain, one observes the formation of a stationary localized solution after transient oscillations (Fig. 5a). With the increase of the delay above the critical value $\tau_c(p)$ shown in Fig. 4, the localized solution becomes unstable with respect to oscillatory instability leading to the formation of an oscillating localized pulse. The formation of such pulse is shown in Fig. 5b. Our numerical simulations confirm the results of the linear stability analysis very well.

We have also found that, with the increase of delay, a localized oscillating pulse is formed near the instability threshold only if the value of the control parameter is large enough, $p > p_* \approx 2.8$. For $p < p_*$ the oscillatory instability is subcritical and leads to blow-up. For $p > p_*$ the instability is supercritical; it results in an oscillating localized pulse, with the oscillation amplitude growing with the growth of the delay near the instability threshold as $(\tau - \tau_c)^{1/2}$, as shown in Fig. 6b. The phase portraits corresponding to the pulse oscillations

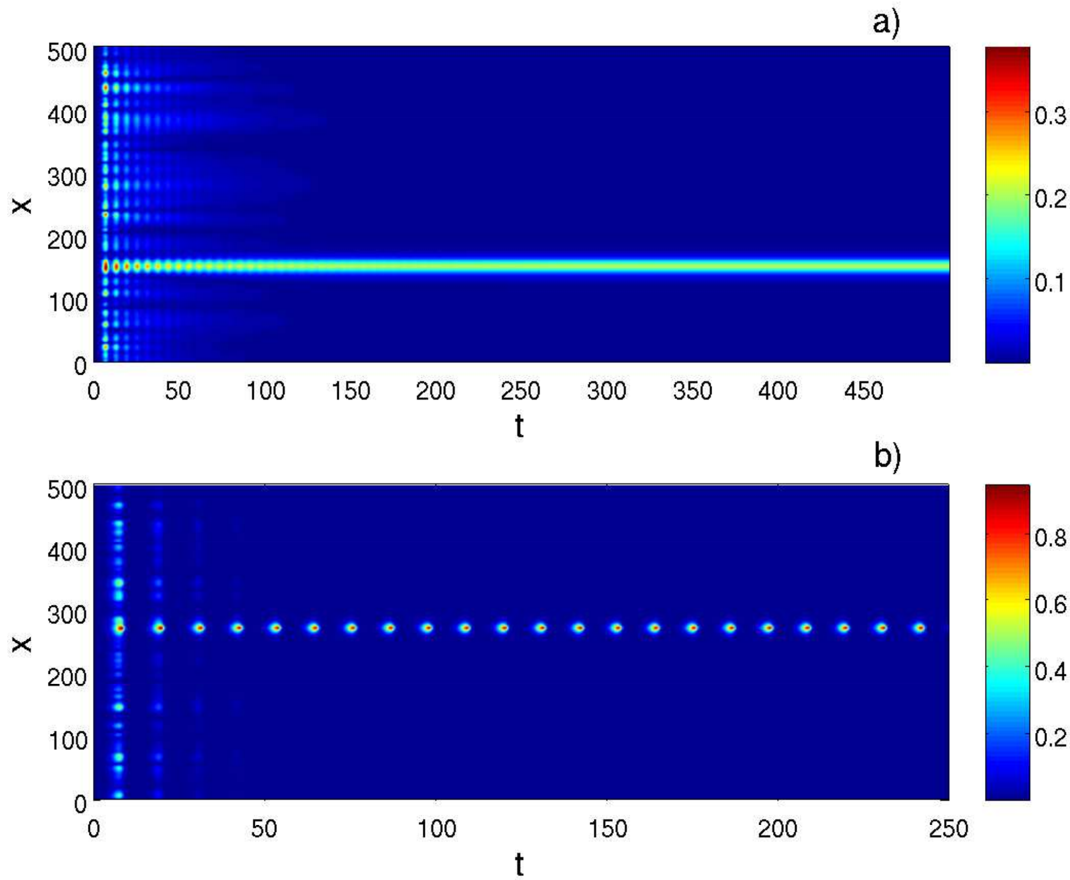

FIG. 5: Spatio-temporal diagrams of numerical solutions of Eq. (3) for $p = 5.0$ and (a) $\tau = 1.4$ (stationary localized solution), (b) $\tau = 1.9$ (oscillating localized pulse).

are shown in Fig. 6a. For a given $p > p_*$, when the delay τ approaches a threshold value τ_*, the solution blows up due to merging of stable and unstable limit cycles. This is also illustrated in Fig. 6b.

V. CONCLUSIONS

We have investigated the dynamics of subcritically unstable pattern forming systems under the action of a delayed global feedback control within the framework of the controlled Ginzburg-Landau equation for the pattern envelope function (3) . The control is based on the measurement of the pattern maximum amplitude. We have shown that the feedback control stabilizes a stationary localized solution of eq. (3) determined by (4)-(6) that leads to formation of spatially localized patterns. The localized solution can exist only if the

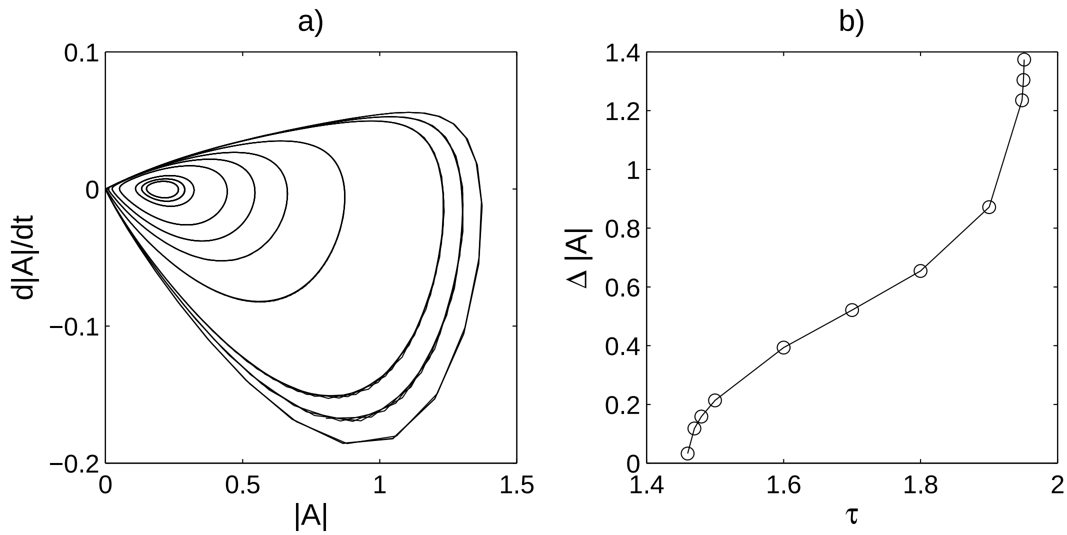

FIG. 6: Results of numerical simulations of Eq. (3) showing: (a) – phase portrait of oscillating localized pulses for $p = 5.0$ and $\tau = 1.47,\ 1.48,\ 1.5,\ 1.6,\ 1.7,\ 1.8,\ 1.9,\ 1.948,\ 1.950,\ 1.951$; the values of $|A|$ correspond to the spatial location of the pulse maximum. (b) – amplitude of an oscillating localized pulse, $\Delta|A|$, as a function of delay τ.

value of the control parameter is larger than $\sqrt{2}$. We have performed a linear stability analysis of this solution and obtained an analytic dispersion relation (28) that determines the perturbation growth rate on the values of the control strength and delay. The linear stability analysis shows that with the increase of delay the stationary localized solution exhibits an oscillatory instability. Stability region in the control strength-delay plane is found. We have shown that this instability can be either subcritical, leading to a blow-up, or supercritical, leading to the formation of spatially-localized oscillating pulses corresponding to localized oscillating patterns. We have performed numerical simulations of Eq. (3) that confirmed these conclusions. By means of numerical simulations we have found a critical value of the control parameter above which the formation of delay-driven localized oscillating pulses is possible. We have also found that with further increase of delay the pulse oscillation amplitude blows-up due to merging of stable and unstable limit cycles.

VI. ACKNOWLEDGEMENTS

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