# Lamb-type solution and properties of unsteady Stokes equations 

<br>${ }^{1}$ Department of Chemical Engineering, Technion, Haifa 32000, Israel<br>${ }^{2}$ Faculty of Mechanical Engineering, Technion - Israel Institute of Technology, Haifa, 3200003, Israel<br>${ }^{3}$ Department of Computational Science and Engineering,<br>Yonsei University, Seoul 120-749, South Korea and<br>${ }^{4}$ Stowers Institute for Medical Research, 1000 E 50th st.,Kansas City, MO 64110, USA


#### Abstract

In the present paper we derive the general solution of the unsteady Stokes equations in an unbounded fluid in spherical polar coordinates. The solution is an expansion in vector spherical harmonics and given as a sum of a particular solution, proportional to pressure gradient exhibiting power-law dependence, and a solution of vector Helmholtz equation decaying exponentially fast at infinity. The proposed decomposition resembles the classical Lamb's solution for the steady Stokes equations: the series coefficients are projections of radial component, divergence and curl of the boundary flow on scalar spherical harmonics. The proposed solution provides an explicit form of the potential far from an oscillating body ("generalized Darcy's law") and high- and low-frequency expansions. The leading order of the high-frequency expansion yields the well-known ideal (inviscid) flow approximation. Continuation of the proposed solution to imaginary frequency provides general solution of the Brinkman equations describing viscous flow in porous medium.


## I. INTRODUCTION

The method of spherical harmonics expansion is a standard method for solving the Laplace equation widely used in the various fields. In this work we derive a similar expansion for the transient Stokes equations describing low-Reynolds-number $(R e<1)$ transient or unsteady flows of incompressible viscous fluid [1-3] . The smallness of Re allows to drop the nonlinear (quadratic) term in the full Navier-Stokes equations yielding the linear unsteady Stokes equations, possessing a general solution. In the present paper we describe the methods of construction of the general solutions of the unsteady Stokes equations and study their properties.

Low-Reynolds-number (viscous) hydrodynamics distinguishes between steady Stokes equations, obtained by dropping all inertia terms (i.e., due to the material derivative of the velocity) in the Navier-Stokes equations, and unsteady Stokes equations obtained by keeping the velocity time-derivative (i.e., the Eulerian acceleration term). The steady Stokes equations read $\eta \nabla^{2} \boldsymbol{v}=\nabla p$, where $\boldsymbol{v}$ is the incompressible (solenoidal) fluid velocity, $p$ is the pressure and $\eta$ is the dynamic viscosity of the fluid; they contain no explicit time dependence and the quasi-static approximation applies. There are several known representations of the general solution of the steady Stokes equations. The seminal Lamb's solution [4] is a sum of three series, whose terms are composed from solid spherical harmonics (see, e.g., 2]). The first series is a particular solution of the Stokes equations due to the source $\nabla p$, while and the remaining two series provide a general solenoidal solution of the vector Laplace

[^0]equation $\nabla^{2} \boldsymbol{v}=0$ (see [1]). Another form of the general solution is given by the so-called adjoint method that uses the expansion of an arbitrary vector field into a complete set of vector functions derived from the spherical harmonics [1, 5]. This set is rather similar to vector spherical harmonics (VSH) employed here (see [6]), however has somewhat more cumbersome orthogonality relations. Other forms of the general solution of the Stokes equations are the Papkovich-Neuber [7] , the NaghdiHsu [10-12] and the Boussinesq-Galerkin [11, 13] formulations. Finally, 14] provided a general solution derived from poloidal-toroidal decomposition of incompressible flow 15, 16] (see also [17]). Lamb's decomposition is by far the most useful, see, e.g., 18] for numerical implementation for many-particle systems.

In striking contrast to the steady Stokes equations, the general solutions of the transient Stokes equations, given in the frequency domain by $-i \rho \omega \boldsymbol{v}=\eta \nabla^{2} \boldsymbol{v}-\nabla p$, attracted far less attention. Here $\rho$ is the fluid density, $\omega$ is the frequency and $i$ is the imaginary unit. The main reason for lesser attention is that typically the time and convective derivative terms are of the same order of magnitude. The transient Stokes equations apply when the flow is periodic and/or has significant time dependence. The general solution for axially symmetric case was provided in [19] in terms of the stream-function. For nonaxisymmetric flows the only available representation (to the best of our knowledge) is the poloidal-toroidal decomposition [20]. This solution is, however, less transparent than the one derived here, as shall be demonstrated below. Our representation of the solution is similar to Lamb's decomposition, i.e., it is constructed as a sum of the particular solution $-i \omega^{-1} \nabla(p / \rho)$ where $\nabla^{2} p=0$, and a general solution of the solenoidal vector Helmholtz equation with an imaginary coefficient, $-i \omega \boldsymbol{v}=\nu \nabla^{2} \boldsymbol{v}$; here $\nu=\eta / \rho$ is the kinematic viscosity. The standard fundamental set of solutions of the vector Helmholtz equation with real-valued coefficients, $-k^{2} \boldsymbol{v}=\nu \nabla^{2} \boldsymbol{v}$, is
provided by the VSH (see, e.g. [21, 22]). The radial dependence of the set's functions is given by spherical Bessel functions. The extension to imaginary coefficient leads to Bessel functions of imaginary argument or modified Bessel functions, which can be reduced to polynomials. The resulting solution of the Helmholtz equation decays exponentially fast away from the origin and is given by the elementary functions. Here we only consider the solution of the exterior problem, or the flow external to a sphere, while the corresponding interior problem was solved in [23].

The proposed representation provides some important insights. For example, it is well known that at distances beyond the viscous penetration depth $\delta=(2 \nu / \omega)^{1 / 2}$ from an oscillating body the flow is potential [3]. This property readily follows from the fact that the vorticity $\zeta \equiv \nabla \times v$ obeys the vector Helmholtz equation $-i \omega \boldsymbol{\zeta}=\nu \nabla^{2} \boldsymbol{\zeta}$, and thus decays exponentially fast away from the object's surface where it is generated (alternatively, this property can be demonstrated by using integral representation of the flow and properties of the fundamental solution [24]). In our approach the emergence of the potential flow is an immediate consequence of the exponential decay of solutions of the Helmholtz equation and a simple, yet fundamental connection between the flow and the pressure, $\boldsymbol{v} \approx-i \omega^{-1} \nabla(p / \rho)$, at distances larger than $\delta$. This result can also be obtained by rewriting the flow equation as $i \omega \boldsymbol{v}-\rho^{-1} \nabla p=\nu \nabla \times \boldsymbol{\zeta}$ and considering the exponential decay of the right-hand-side (RHS) far from the body. It seems that both representations derived here, $\boldsymbol{v} \approx-i \omega^{-1} \nabla(p / \rho)$ and the expression for $p$ via the boundary conditions, are missing in the literature.

The limit of an ideal (or inviscid) flow is a fundamental topic in fluid mechanics [3, 25]. In this limit $\delta$ tends to zero and the above considerations imply that $\boldsymbol{v} \approx-i \omega^{-1} \nabla(p / \rho)$ hold everywhere, in accord with the ideal flow approximation. Within this approximation, the potential $-i \omega^{-1} p / \rho$ is obtained as the solution of the Laplace equation, whose normal derivative at the surface coincides with the corresponding component of the velocity at the boundary (see, e.g., [24, 26]). We are not aware of a rigorous proof of this representation of the potential. We show here that the ideal flow approximation is the leading term of the expansion of the general solution in the viscosity coefficient. The expansion parameter is $\sqrt{\nu}$, rather than $\nu$, indicating that adding viscosity yields a singular perturbation. The dimensionless expansion parameter is $1 / \sqrt{R o}$ where $R o=a^{2} \omega / \nu$ is the Roshko number defined with the radius $a$ of the sphere at which the boundary conditions are prescribed. This asymptotic series can alternatively be considered as the high-frequency expansion.

The Roshko number, Ro, provides an estimate for the ratio of the Eulerian time-derivative and viscous terms in the unsteady Stokes equations and can be written as product of the Strouhal number, $S l=t_{s} / T$, and Reynolds number, $R e=v_{0} a / \nu$. Here $t_{s}=a / v_{0}$ is the Stokes time with $v_{0}$ being a characteristic velocity and
$T$ is the characteristic time $1 / \omega$. If $R o \ll 1$, then the quasi-static (low-frequency) approximation provided by the Stokes equations applies. We demonstrate below that both the Lamb's and the adjoint method's solutions of the Stokes equation can be obtained in the limit $R o \rightarrow 0$. However we demonstrate that care needs to be exercised when applying the approximation. Corrections to the low-frequency quasi-static approximation are given by a series expansion in $\sqrt{R o}$. Therefore, similarly to the high-frequency expansion, low-frequency expansion also proves to be singular.

We observe that $\boldsymbol{v} \approx-i \omega^{-1} \nabla(p / \rho)$ that holds at large distances is remarkably similar to the Darcy's law for porous medium. In fact, it can be called the generalized Darcy's law because it constitutes the analytic continuation of that law as we demonstrate in Sec. IX, The demonstration is done by providing the general solution of the Brinkman equations [27, 28].

We believe that our work is a significant step toward general understanding of the unsteady Stokes equations and several important closely related topics in viscous hydrodynamics. Potential applications are considered in the Conclusions section.

## II. GENERAL SOLUTION OF UNSTEADY STOKES EQUATIONS

In this work we derive the general solution of unsteady Stokes equations

$$
\begin{equation*}
\partial_{t} \boldsymbol{v}=-\rho^{-1} \nabla p+\nu \nabla^{2} \boldsymbol{v}, \quad \nabla \cdot \boldsymbol{v}=0 \tag{1}
\end{equation*}
$$

in spherical coordinates. We use the Fourier transform

$$
\begin{equation*}
\boldsymbol{v}(\omega, \boldsymbol{x}) \equiv \int \boldsymbol{v}(t, \boldsymbol{x}) \mathrm{e}^{i \omega t} d t \tag{2}
\end{equation*}
$$

where we use the same letter for the Fourier-transformed variable with no ambiguity. The Fourier-transformed flow equations in the frequency domain then read

$$
\begin{equation*}
-i \omega \boldsymbol{v}=-\rho^{-1} \nabla p+\nu \nabla^{2} \boldsymbol{v}, \quad \nabla \cdot \boldsymbol{v}=0 \tag{3}
\end{equation*}
$$

The study of the above equation is performed assuming $\omega>0$. Once the solution is obtained we find using $\boldsymbol{v}(-\omega)=\boldsymbol{v}^{*}(\omega)$ that

$$
\begin{equation*}
\boldsymbol{v}(t, \boldsymbol{x})=\int_{0}^{\infty} \boldsymbol{v}(\omega, \boldsymbol{x}) \mathrm{e}^{-i \omega t} \frac{d \omega}{2 \pi}+c . c . \tag{4}
\end{equation*}
$$

where c.c. stands for complex conjugate. We shall study the solution in the exterior of a sphere with radius $a$. The solution for the interior problem was provided in 23] and has quite different properties due to the absence of farflow regime. We assume that the flow vanishes at infinity and is prescribed at the sphere surface at $r=a$ where $r$ is the radial coordinate.

Solutions with given frequency $\omega$ can be characterized by the viscous penetration depth, $\delta=\sqrt{2 \nu / \omega}$. It provides the characteristic length of penetration of vorticity generated at the boundaries, into the fluid bulk, see e.g. [3].

The strength of the frequency term with respect to the viscosity term in Eqs. (2) is determined by the dimensionless Roshko number, $R o \equiv a^{2} \omega / \nu$, estimating the terms' ratio, see the Introduction. The Stokes equations' limit holds at $R o \rightarrow 0$. The condition of negligibility of the frequency term is $\sqrt{R o} \ll 1$ and not $R o \ll 1$, similarly to the fundamental solution [1], see below. Frequency term is a singular perturbation of the Stokes equations [24]. For $R o \gtrsim 1$ the unsteady term cannot be neglected.

Dimensionless variables-We will use dimensionless variables by measuring velocity in units of $a \omega$, pressure in units of $\eta \omega$ and coordinate in units of $a$. We designate dimensionless velocity by $\boldsymbol{u}$ and the dimensionless pressure by $p$ (we use the same letter with no ambiguity below). We have

$$
\begin{equation*}
\lambda^{2} \boldsymbol{u}=-\nabla p+\nabla^{2} \boldsymbol{u}, \quad \nabla \cdot \boldsymbol{u}=0 ; \quad \lambda^{2}=-i R o=-\frac{i a^{2} \omega}{\nu} \tag{5}
\end{equation*}
$$

We define below the square root by $\lambda=(1-i) \sqrt{R o / 2}$. In the limit $R o \rightarrow 0$ Eqs. (5) reduce to steady Stokes equations and for $R o \gtrsim 1$ the unsteady term in the equations cannot be neglected (more precisely the neglect is invalid at $\sqrt{R o} \gtrsim 1$, see below). We look below for the general solution of Eqs. (5) in spherical coordinates. The solution involves arbitrary constants that can then be fixed once the values of $\boldsymbol{u}$ on the sphere $r=1$ are considered as prescribed.

General form of the pressure - We observe by taking divergence of the first of Eqs. (5) that pressure is a harmonic function. Therefore it can be represented as,

$$
\begin{equation*}
p(\boldsymbol{r})=\sum_{l m} \frac{c_{l m} Y_{l m}(\theta, \phi)}{r^{l+1}}, \text { where } \sum_{l m} \equiv \sum_{l=1}^{\infty} \sum_{m=-l}^{l}, \tag{6}
\end{equation*}
$$

where we used the boundary conditions at infinity and the spherical harmonics $Y_{l m}(\theta, \phi)$ are defined by

$$
\begin{equation*}
Y_{l m}=\sqrt{\frac{(2 l+1)}{4 \pi} \frac{(l-m)!}{(l+m)!}} P_{l}^{m}(\cos \theta) \exp (i m \phi) \tag{7}
\end{equation*}
$$

where $P_{l}^{m}$ are associated Legendre polynomials. We use the multiplicative factor of [6] in the definition of $Y_{l m}(\theta, \phi)$, so the following orthogonality relation among the spherical harmonics reads
$\int Y_{l m} Y_{l^{\prime} m^{\prime}}^{*} d \Omega=\int_{0}^{\pi} \sin \theta d \theta \int_{0}^{2 \pi} d \phi Y_{l m} Y_{l^{\prime} m^{\prime}}^{*}=\delta_{l l^{\prime}} \delta_{m m^{\prime}}$. The term with $l=0$ is omitted in Eq. (6) assuming that there is no net mass flux to infinity (see the next subsection). The constant coefficients $c_{l m}$ are to be determined from the boundary conditions on the velocity. The coefficients depend on $\lambda$ as a parameter in the equation.

## A. Qualitative behavior of the solution as superposition of power-law and exponentially decaying terms

A particular solution of Eqs. (5) for $\boldsymbol{u}$, where $p$ is considered as a source, is $-\nabla p / \lambda^{2}$. We observe that the
solution diverges at $\omega \rightarrow 0$ and does not reduce to the similar Lamb's partial solution of the Stokes equations, see 1] and below. Still the use of this particular solution in comparison to others is advantageous, because it provides a straightforward decomposition of $\boldsymbol{u}$ into two terms that have qualitatively different spatial behavior. Indeed the general solution of Eqs. (5) can be constructed as superposition of $-\nabla p / \lambda^{2}$ and a solenoidal solution $\boldsymbol{u}_{s}$ of vector Helmholtz equation

$$
\begin{equation*}
\boldsymbol{u}=\boldsymbol{u}_{s}-\frac{\nabla p}{\lambda^{2}}, \quad \lambda^{2} \boldsymbol{u}_{s}=\nabla^{2} \boldsymbol{u}_{s}, \quad \nabla \cdot \boldsymbol{u}_{s}=0 \tag{8}
\end{equation*}
$$

Solutions of the Helmholtz equation are known to decay exponentially away from the source with exponent given by real part of $\lambda$ (e.g. the asymptotic behavior of solutions of at large distances $r$ is given by the exponential decay $\exp (-\lambda r))$. Thus, considering the boundary conditions on the sphere as the source, $\boldsymbol{u}_{s}$ component of the solution is negligible at distances $r-1$ from the sphere that are much larger than the dimensionless penetration depth $\delta / a$ (we observed that the real part of $\lambda$ is $\sqrt{R o / 2}=a / \delta)$. The flow there is given by $\boldsymbol{u} \approx-\nabla p / \lambda^{2}$. It is potential and decays as a power law, see Eq. (6) and below. It is remarkable that these conclusions can be arrived at via qualitative study of the solutions. We remark that inclusion of $l=0$ term in Eq. (6) would lead to finite flux $\propto \nabla r^{-1}$ behavior of the flow at infinity.

## B. Vector spherical harmonics

To complete the general solution of unsteady Stokes equations one has to determine the general solution of the vector Helmholtz equation, see Eqs. (6) and (8). This is often achieved by using the vector spherical harmonics (VSH) which generalize the ordinary harmonics to vector case 21, 22, 39]. Easy to use definition of the VSH was given in [6],

$$
\begin{align*}
& \boldsymbol{Y}_{l m}=\hat{\boldsymbol{r}} Y_{l m}, \quad \mathbf{\Psi}_{l m}=r \nabla Y_{l m}=\hat{\boldsymbol{\theta}} \partial_{\theta} Y_{l m}+\frac{\hat{\boldsymbol{\phi}} \partial_{\phi} Y_{l m}}{\sin \theta} \\
& \mathbf{\Phi}_{l m}=\boldsymbol{r} \times \nabla Y_{l m}=-\nabla \times\left(\boldsymbol{r} Y_{l m}\right)=\hat{\boldsymbol{\phi}} \partial_{\theta} Y_{l m}-\frac{\hat{\boldsymbol{\theta}} \partial_{\phi} Y_{l m}}{\sin \theta} \tag{9}
\end{align*}
$$

where $\nabla$ is the three-dimensional gradient and $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are the unit vectors of the spherical coordinate system. For instance,

$$
\begin{equation*}
Y_{10}=\sqrt{\frac{3}{4 \pi}} \cos \theta, \quad \boldsymbol{\Psi}_{10}=-\hat{\theta} \sqrt{\frac{3}{4 \pi}} \sin \theta \tag{10}
\end{equation*}
$$

and we have that

$$
\begin{equation*}
-\nabla\left(\frac{Y_{l m}}{r^{l+1}}\right)=\frac{(l+1) \boldsymbol{Y}_{l m}-\boldsymbol{\Psi}_{l m}}{r^{l+2}} \tag{11}
\end{equation*}
$$

Thus the particular solution's component of the general solution $-\nabla p / \lambda^{2}$ obeys

$$
\begin{equation*}
-\frac{\nabla p}{\lambda^{2}}=\sum_{l m} c_{l m} \frac{(l+1) \boldsymbol{Y}_{l m}-\boldsymbol{\Psi}_{l m}}{\lambda^{2} r^{l+2}} \tag{12}
\end{equation*}
$$

Any vector field can be represented as a series in dimensionless VSH [6]. We introduce the expansion of $\boldsymbol{u}_{s}$

$$
\begin{equation*}
\boldsymbol{u}_{s}=\sum_{l m}\left(c_{l m}^{r}(r) \boldsymbol{Y}_{l m}+c_{l m}^{(1)}(r) \boldsymbol{\Psi}_{l m}+c_{l m}^{(2)}(r) \boldsymbol{\Phi}_{l m}\right) \tag{13}
\end{equation*}
$$

where $c_{l m}^{r}(r), c_{l m}^{(i)}(r)$ are certain functions of the radial variable $r$ only. The $l=0$ term is omitted since $\boldsymbol{\Psi}_{00}$ and $\boldsymbol{\Phi}_{00}$ vanish identically and $c_{00}^{r}=0$, see below and Appendix A .

## C. General solution

We solve the Helmholtz equation for $\boldsymbol{u}_{s}$ by the plugging into it the expansion of $\boldsymbol{u}_{s}$ in the VSH and solving the resulting system of linear ordinary differential equations for $c_{l m}^{r}(r), c_{l m}^{(i)}(r)$. The solution is presented in Appendix A. The radial coefficient functions can be written via elementary functions producing the general solution

$$
\begin{align*}
& \boldsymbol{u}_{s}=\frac{\exp (-\lambda r)}{r} \sqrt{\frac{\pi}{2 \lambda}} \sum_{l m}\left(\sum_{k=0}^{l} \frac{(l+k)!}{k!(l-k)!(2 \lambda r)^{k}}\right. \\
& \times\left(\frac{\tilde{c}_{l m}^{r} \boldsymbol{Y}_{l m}}{r}+\tilde{c}_{l m} \boldsymbol{\Phi}_{l m}\right)-\frac{\tilde{c}_{l m}^{r} \boldsymbol{\Psi}_{l m}}{(l+1) r}\left(\sum_{k=0}^{l} \frac{(l+k)!}{k!(l-k)!(2 \lambda r)^{k}}\right. \\
& \left.\left.+\frac{1}{2 l} \sum_{k=0}^{l-1} \frac{(l-1+k)!}{k!(l-1-k)!(2 \lambda r)^{k-1}}\right)\right) . \tag{14}
\end{align*}
$$

The solution is defined by two sets of constant coefficients $\tilde{c}_{l m}^{r}$ and $\tilde{c}_{l m}$ (one out of three sets for vector solutions is not showing due to solenoidality). It can be observed that the solution is given by $\exp (-\lambda r)$ times a series which looks quite similar to the solution of the Laplace equation, apart from the fact that each $l m$ term involves a polynomial $\mathcal{P}_{l}\left(\lambda^{-1} r^{-1}\right)$ in negative integer powers of $r$ and not just a single power. We defined

$$
\begin{equation*}
\mathcal{P}_{l}(x) \equiv \sum_{k=0}^{l} \frac{(l+k)!x^{k}}{k!(l-k)!2^{k}} \tag{15}
\end{equation*}
$$

This polynomial originates from the modified Bessel functions $K_{\nu}(x)$ encountered in the solution of the Helmholtz equation with imaginary coefficient,

$$
\begin{equation*}
K_{l+1 / 2}(x)=\sqrt{\frac{\pi}{2 x}} e^{-x} \mathcal{P}_{l}\left(\frac{1}{x}\right) \tag{16}
\end{equation*}
$$

Thus elementary solutions $\chi_{l m}^{\lambda}$ of the scalar Helmholtz equation which vanish at infinity are

$$
\begin{equation*}
\chi_{l m}^{\lambda} \equiv-\frac{K_{l+1 / 2}(\lambda r) Y_{l m}}{\sqrt{r}}, \quad \nabla^{2} \chi_{l m}^{\lambda}=\lambda^{2} \chi_{l m}^{\lambda} \tag{17}
\end{equation*}
$$

We can rewrite Eq. (14) as

$$
\begin{align*}
& \boldsymbol{u}_{s}=\sum_{l m}\left(\frac{\tilde{c}_{l m}^{r} K_{l+1 / 2}(\lambda r) \boldsymbol{Y}_{l m}}{r^{3 / 2}}+\frac{\tilde{c}_{l m} K_{l+1 / 2}(\lambda r) \boldsymbol{\Phi}_{l m}}{\sqrt{r}}\right. \\
& \left.-\frac{\tilde{c}_{l m}^{r} \boldsymbol{\Psi}_{l m}}{l(l+1) r}\left(\frac{l K_{l+1 / 2}(\lambda r)}{\sqrt{r}}+\lambda r^{1 / 2} K_{l-1 / 2}(\lambda r)\right)\right) \cdot \tag{18}
\end{align*}
$$

The Bessel functions are also showing in the general solution of the unsteady Stokes equations in the axially symmetric [19] and general 20] cases. The use of $\mathcal{P}_{l}(x)$ emphasizes that the solution is given in terms of the elementary functions and does not involve fractional powers of $r$. (We found the relation between modified Bessel functions of half-integer order and Bell polynomials [29] less useful.) The full solution, given by superposition in Eq. (8) reads

$$
\begin{align*}
& \boldsymbol{u}=\sum_{l m}\left(\left(\frac{\tilde{c}_{l m}^{r} K_{l+1 / 2}(\lambda r)}{r^{3 / 2}}+\frac{(l+1) c_{l m}}{\lambda^{2} r^{l+2}}\right) \boldsymbol{Y}_{l m}\right. \\
& +\frac{\tilde{c}_{l m} K_{l+1 / 2}(\lambda r)}{\sqrt{r}} \boldsymbol{\Phi}_{l m}-\boldsymbol{\Psi}_{l m}\left(\frac{c_{l m}}{\lambda^{2} r^{l+2}}\right. \\
& \left.\left.+\frac{\tilde{c}_{l m}^{r}}{l(l+1) r}\left(\frac{l K_{l+1 / 2}(\lambda r)}{\sqrt{r}}+\lambda r^{1 / 2} K_{l-1 / 2}(\lambda r)\right)\right)\right) . \tag{19}
\end{align*}
$$

We shall now proceed to calculation of the coefficients.

## D. Coefficients of expansion

The free constants $c_{l m}, \tilde{c}_{l m}^{r}$ and $\tilde{c}_{l m}$ are to be determined from the boundary conditions. We consider the case when these are given by prescribed values of velocity on the unit sphere. We find by projecting the general solution given by Eq. (19) onto the VSH and using the VSH orthogonality relations of [6] that the constant coefficients obey

$$
\begin{align*}
& \int u_{r} Y_{l m}^{*} d \Omega=\tilde{c}_{l m}^{r} K_{l+1 / 2}(\lambda)+\frac{(l+1) c_{l m}}{\lambda^{2}}  \tag{20}\\
& \int \boldsymbol{u} \cdot \mathbf{\Psi}_{l m}^{*} d \Omega=-\tilde{c}_{l m}^{r}\left(l K_{l+1 / 2}(\lambda)+\lambda K_{l-1 / 2}(\lambda)\right) \\
& -\frac{c_{l m} l(l+1)}{\lambda^{2}}, \quad \int \boldsymbol{u} \cdot \boldsymbol{\Phi}_{l m}^{*} d \Omega=l(l+1) \tilde{c}_{l m} K_{l+1 / 2}(\lambda),
\end{align*}
$$

where we used $\boldsymbol{u} \cdot \boldsymbol{Y}_{l m}^{*}=u_{r} Y_{l m}^{*}$. Multiplying the first of the above equations by $l$ and combining it with the second equation gives

$$
\begin{equation*}
\tilde{c}_{l m}^{r}=-\frac{l \int u_{r} Y_{l m}^{*} d \Omega+\int \boldsymbol{u} \cdot \mathbf{\Psi}_{l m}^{*} d \Omega}{\lambda K_{l-1 / 2}(\lambda)} . \tag{21}
\end{equation*}
$$

We thus find that

$$
\begin{align*}
& c_{l m}=\frac{\lambda\left(l K_{l+1 / 2}(\lambda)+\lambda K_{l-1 / 2}(\lambda)\right)}{(l+1) K_{l-1 / 2}(\lambda)} \int u_{r} Y_{l m}^{*} d \Omega \\
& +\frac{\lambda K_{l+1 / 2}(\lambda)}{(l+1) K_{l-1 / 2}(\lambda)} \int \boldsymbol{u} \cdot \mathbf{\Psi}_{l m}^{*} d \Omega \tag{22}
\end{align*}
$$

It is useful in calculations to observe the simple relation between the coefficients $c_{l m}$ and $\tilde{c}_{l m}^{r}$

$$
\begin{equation*}
c_{l m}=\frac{\lambda^{2}}{l+1} \int u_{r} Y_{l m}^{*} d \Omega-\frac{\lambda^{2} K_{l+1 / 2}(\lambda) \tilde{c}_{l m}^{r}}{l+1} \tag{23}
\end{equation*}
$$

Eqs. (20)-(22) provide the coefficients of the expansion via projections of the surface velocity onto the VSH.

These projections involve vectors and their direct derivation is cumbersome. The calculations are simplified by using the following identities

$$
\begin{align*}
& \int Y_{l m}^{*} \nabla_{s} \cdot \boldsymbol{u} d \Omega=2 \int Y_{l m}^{*} u_{r} d \Omega-\int \boldsymbol{u} \cdot \mathbf{\Psi}_{l m}^{*} d \Omega \\
& \int_{r=1} Y_{l m}^{*}(\nabla \times \boldsymbol{u})_{r} d \Omega=-\int \boldsymbol{u} \cdot \mathbf{\Phi}_{l m}^{*} d \Omega \tag{24}
\end{align*}
$$

derived in Appendix B Above $\nabla_{s} \cdot \boldsymbol{u}$ is known as surface divergence [1] at $r=1$

$$
\begin{equation*}
\nabla_{s} \cdot \boldsymbol{u}=\nabla \cdot \boldsymbol{u}-\frac{\partial u_{r}}{\partial r}=2 u_{r}+\frac{\partial_{\theta}\left(\sin \theta u_{\theta}\right)+\partial_{\phi} u_{\phi}}{\sin \theta} \tag{25}
\end{equation*}
$$

The first form in the above equation involves $\partial_{r} u_{r}(r=1)$ that is not provided by the boundary conditions. This value can be obtained by using any continuation of the boundary condition to $r \neq 1$ since the end result is independent of that continuation, as seen from the second form. The utility of the first form is due to the fact that in actual calculations a simple continuation of the boundary conditions is often evident, see examples below. We notice that the radial component of the curl of the flow is determined uniquely by $\boldsymbol{u}(r=1)$ so the integrals in Eqs. (24) are uniquely determined by the flow at the surface. The above formulae allow to find the coefficients of the solution by projecting the scalar functions onto the usual spherical harmonics. We have for the pressure coefficients $c_{l m}$, defined in Eq. (6), that

$$
\begin{align*}
& c_{l m}=\frac{\lambda\left((l+2) \mathcal{P}_{l}\left(\lambda^{-1}\right)+\lambda \mathcal{P}_{l-1}\left(\lambda^{-1}\right)\right)}{(l+1) \mathcal{P}_{l-1}\left(\lambda^{-1}\right)} \int_{r=1} Y_{l m}^{*} u_{r} d \Omega \\
& -\frac{\lambda \mathcal{P}_{l}\left(\lambda^{-1}\right)}{(l+1) \mathcal{P}_{l-1}\left(\lambda^{-1}\right)} \int Y_{l m}^{*} \nabla_{s} \cdot \boldsymbol{u} d \Omega \tag{26}
\end{align*}
$$

Similarly we find

$$
\begin{equation*}
\tilde{c}_{l m}=-\frac{1}{l(l+1) K_{l+1 / 2}(\lambda)} \int_{r=1} Y_{l m}^{*}(\nabla \times \boldsymbol{u})_{r} d \Omega \tag{27}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{c}_{l m}^{r}=\frac{\int Y_{l m}^{*} \nabla_{s} \cdot \boldsymbol{u} d \Omega-(l+2) \int u_{r} Y_{l m}^{*} d \Omega}{\lambda K_{l-1 / 2}(\lambda)} \tag{28}
\end{equation*}
$$

The above formulas show that the complexity in finding the coefficients of the expansion of the unsteady Stokes equation is the same as of the usual Stokes equations the expansion of Lamb's solution is determined by the same projections of the boundary flow [1].

## III. LAMB-TYPE FORM OF THE SOLUTION

In this section, we show that Lamb's solution of the steady Stokes equations $\nabla p=\nabla^{2} \boldsymbol{u}$, that was originally provided without derivation [4], could be obtained in a way similar to our approach above. Again pressure is a harmonic function that for solutions vanishing at infinity can be written as $p=\sum_{l=1}^{\infty} p_{-l-1}$. Here $p_{-l-1}$ is
a solid spherical harmonics given by linear combination of $r^{-l-1} Y_{l m}$ with $m$ ranging from $-l$ to $l$, see Eq. (6) and [1]. A partial, solenoidal solution of the Stokes equations $\nabla^{2} \boldsymbol{u}_{n}=\nabla p_{n}$ is $\alpha_{n}\left(r^{2} \nabla p_{n}-2 n \boldsymbol{r} p_{n} /(n+3)\right)$ where $2 \alpha_{n}(2 n+3)(n+1)=n+3$, see exercise 4.1 in [1]. The general solution for the steady Stokes flow vanishing at $r=\infty$ is then written as

$$
\begin{align*}
& \boldsymbol{u}^{L a m b}=\sum_{l=1}^{\infty}\left(-\frac{(l-2) r^{2} \nabla p_{-l-1}}{2 l(2 l-1)}+\frac{(l+1) \boldsymbol{r} p_{-l-1}}{l(2 l-1)}\right) \\
& +\sum_{l=1}^{\infty}\left(\nabla \Phi_{-l-1}+\nabla \times\left(\boldsymbol{r} \chi_{-l-1}\right)\right) \tag{29}
\end{align*}
$$

Here the first line is a partial solution of $\nabla^{2} \boldsymbol{u}=\nabla p$ and the last line provides a form of the general solenoidal solution of the vector Laplace equation $\nabla^{2} \boldsymbol{u}=0$ with $\Phi_{-l-1}, \chi_{-l-1}$ solid spherical harmonics similar to $p_{-l-1}$, see [1].

Generalization of Lamb's solution to finite frequency-A main result of our work is that our general solution can be written in the form similar to Lamb's

$$
\begin{equation*}
\boldsymbol{u}=-\frac{\nabla p}{\lambda^{2}}+e^{\lambda(1-r)} \boldsymbol{u}^{H}+\nabla \times\left(\boldsymbol{r} e^{\lambda(1-r)} X\right) \tag{30}
\end{equation*}
$$

where $\boldsymbol{u}^{H}$ and $X$ are certain vector and scalar fields, respectively. Similarly to Lamb's solution the first term is a partial solution and the last two terms are solutions of homogeneous equation where one of the terms is toroidal.

Toroidal component-We observe that using the definition $\boldsymbol{\Phi}_{l m}=-\nabla \times\left(\boldsymbol{r} Y_{l m}\right)$ we can write

$$
\begin{equation*}
\sum_{l m} \frac{\tilde{c}_{l m} K_{l+1 / 2}(\lambda r)}{\sqrt{r}} \boldsymbol{\Phi}_{l m}=\sum_{l m} \tilde{c}_{l m} \nabla \times\left(\boldsymbol{r} \chi_{l m}^{\lambda}\right) \tag{31}
\end{equation*}
$$

where we used $\chi_{l m}^{\lambda}$ defined in Eq. (17). The last term provides simple generalization of the last term in Lamb's solution to unsteady Stokes equations. We remark that elementary solutions $\chi_{l m}^{\lambda}$ of the scalar Helmholtz equation generate elementary solenoidal solutions $\nabla \times\left(\boldsymbol{r} \chi_{l m}^{\lambda}\right)$ of the vector Helmholtz equation so the showing of the toroidal component in the solution seems inevitable.

Using the above identity we can introduce the decomposition of $\boldsymbol{u}_{s}$ in Eqs. (8), (18) as

$$
\begin{equation*}
\boldsymbol{u}_{s}=\nabla \times\left(\boldsymbol{r} e^{\lambda(1-r)} X\right)+e^{\lambda(1-r)} \boldsymbol{u}^{H} \tag{32}
\end{equation*}
$$

where the toroidal component of the solution $X$ and $\boldsymbol{u}^{H}$ are defined by
$X \equiv e^{\lambda(r-1)} \sum_{l m} \tilde{c}_{l m} \chi_{l m}^{\lambda} ; \quad \boldsymbol{u}^{H} \equiv e^{\lambda(r-1)} \sum_{l m} \frac{\tilde{c}_{l m}^{r}}{r^{3 / 2}}$
$\left(K_{l+1 / 2}(\lambda r) \boldsymbol{Y}_{l m}-\frac{\left(l K_{l+1 / 2}(\lambda r)+\lambda r K_{l-1 / 2}(\lambda r)\right) \boldsymbol{\Psi}_{l m}}{l(l+1)}\right)$.
The superscript $H$ refers to $\exp (\lambda(1-r)) \boldsymbol{u}^{H}$ being a solenoidal solution of the vector Helmholtz equation,
$\nabla^{2}\left(\exp (\lambda(1-r)) \boldsymbol{u}^{H}\right)=\lambda^{2} \exp (\lambda(1-r)) \boldsymbol{u}^{H}$. We observe from Eqs. (27) and (16) that $X$ can be written as

$$
\begin{equation*}
X=\sum_{l m} \frac{c_{l m}^{t}}{\mathcal{P}_{l}\left(\lambda^{-1}\right)} \mathcal{P}_{l}\left(\frac{1}{\lambda r}\right) \frac{Y_{l m}(\theta, \phi)}{r} \tag{33}
\end{equation*}
$$

where the toroidal field's coefficients $c_{l m}^{t}$ are independent of the frequency

$$
\begin{equation*}
c_{l m}^{t}=\frac{1}{l(l+1)} \int_{r=1} Y_{l m}^{*}(\nabla \times \boldsymbol{u})_{r} d \Omega \tag{34}
\end{equation*}
$$

We observe from Eq. (15) that at $\lambda \rightarrow 0$ we have $\mathcal{P}_{l}\left((\lambda r)^{-1}\right) / \mathcal{P}_{l}\left(\lambda^{-1}\right)=r^{-l}$. Thus $X$ has regular behavior in the limit of small $\lambda$ which reproduces the Lamb's toroidal term,

$$
\begin{equation*}
X(\lambda=0)=\sum_{l m} \frac{c_{l m}^{t} Y_{l m}(\theta, \phi)}{r^{l+1}} \tag{35}
\end{equation*}
$$

Thus the toroidal field in Eq. (32) can be considered as analytic continuation of the corresponding term in Lamb's solution to finite frequency. This term is associated with oscillatory rotations in the boundary condition, see the solution for oscillatory rotation of a rigid sphere below.

Recovering Lamb's solution from finite frequency solution-In contrast with the toroidal term, the first two terms in the RHS of Eq. (30) are not analytic continuation to finite frequency of the corresponding terms in Lamb's solution, see below. For instance the partial solution, given by the first term in the RHS of Eq. (30), diverges at $\lambda \rightarrow 0$ instead of converging to Lamb's partial solution. We demonstrate however that the sum of the first two terms in Eq. (30) converges to the sum of the first two terms in Eq. (29). We have using the definition of $\boldsymbol{u}^{H}$ above and Eqs. (6) and (23) for $p$ that

$$
\begin{align*}
& e^{\lambda(1-r)} \boldsymbol{u}^{H}-\frac{\nabla p}{\lambda^{2}}=-\nabla \sum_{l m} \frac{Y_{l m}(\theta, \phi) \int u_{r} Y_{l m}^{*} d \Omega}{(l+1) r^{l+1}} \\
& +\sum_{l m} C_{l m}^{r}\left(\frac{\Delta(r) Y_{l m}(\theta, \phi) \hat{\boldsymbol{r}}}{\lambda K_{l-1 / 2}(\lambda) r^{l+2}}-\frac{\nabla Y_{l m}(\theta, \phi)}{K_{l-1 / 2}(\lambda)}\right. \\
& \left.\times\left(\frac{\Delta(r)}{\lambda(l+1) r^{l+1}}-\frac{r^{1 / 2} K_{l-1 / 2}(\lambda r)}{l(l+1)}\right)\right), \tag{36}
\end{align*}
$$

where $\nabla$ is the three dimensional gradient operator. We introduced $\Delta(r) \equiv r^{l+1 / 2} K_{l+1 / 2}(\lambda r)-K_{l+1 / 2}(\lambda)$ that vanishes at the surface $r=1$ and rescaled coefficients $C_{l m}^{r} \equiv \tilde{c}_{l m}^{r} \lambda K_{l-1 / 2}(\lambda)$ that by Eq. (28) obey

$$
\begin{equation*}
C_{l m}^{r}=\int Y_{l m}^{*} \nabla_{s} \cdot \boldsymbol{u} d \Omega-(l+2) \int u_{r} Y_{l m}^{*} d \Omega \tag{37}
\end{equation*}
$$

We consider the zero frequency limit of Eq. (140). We use the first terms of the Taylor expansion

$$
\begin{align*}
& \frac{\lambda K_{l+1 / 2}(\lambda r)}{K_{l-1 / 2}(\lambda)}=\frac{\lambda \exp (\lambda(1-r))}{\sqrt{r} \mathcal{P}_{l-1}\left(\lambda^{-1}\right)} \mathcal{P}_{l}\left(\frac{1}{\lambda r}\right)=\frac{2 l-1}{r^{l+1 / 2}} \\
& +\frac{\lambda^{2}}{2}\left(\frac{2 l-1}{(2 l-3) r^{l+1 / 2}}-\frac{1}{r^{l-3 / 2}}\right)+o(\lambda) \tag{38}
\end{align*}
$$

where the derivatives of the function on the LHS at $\lambda=0$ are obtained by observing that the identity
$\left(\frac{\lambda K_{l+1 / 2}(\lambda r)}{K_{l-1 / 2}(\lambda)}\right)_{\lambda}=\frac{\lambda K_{l+1 / 2}(\lambda r) K_{l-3 / 2}(\lambda)}{K_{l-1 / 2}^{2}(\lambda)}-\frac{\lambda r K_{l-1 / 2}(\lambda r)}{K_{l-1 / 2}(\lambda)}$,
that can be proved by using the formulas for derivatives of the modified Bessel functions. We find that

$$
\begin{align*}
& e^{\lambda(1-r)} \boldsymbol{u}^{H}-\frac{\nabla p}{\lambda^{2}}=\sum_{l m}\left(-\nabla \frac{Y_{l m}(\theta, \phi) \int u_{r} Y_{l m}^{*} d \Omega}{(l+1) r^{l+1}}\right. \\
& \left.+\frac{C_{l m}^{r}}{2 r^{l}}\left(\frac{\hat{\boldsymbol{r}} Y_{l m}}{r^{2}}-\hat{\boldsymbol{r}} Y_{l m}-\frac{\nabla Y_{l m}}{(l+1) r}+\frac{(l-2) r \nabla Y_{l m}}{l(l+1)}\right)\right) \tag{39}
\end{align*}
$$

We observe that the pressure representation $p=$ $\sum_{l=1}^{\infty} p_{-l-1}$ (valid at any frequency) where $p_{-l-1}=$ $r^{-l-1} \sum_{m=-l}^{m=l} c_{l m} Y_{l m}$, see Eq. (6), gives at $\lambda=0$

$$
\begin{align*}
& \sum_{l=1}^{\infty}\left(-\frac{(l-2) r^{2} \nabla p_{-l-1}}{2 l(2 l-1)}+\frac{(l+1) \boldsymbol{r} p_{-l-1}}{l(2 l-1)}\right) \\
& =\sum_{l m} C_{l m}^{r}\left(\frac{(l-2) \nabla Y_{l m}}{2 l(l+1) r^{l-1}}-\frac{\hat{\boldsymbol{r}} Y_{l m}}{2 r^{l}}\right) \tag{40}
\end{align*}
$$

where we used that Eq. (23) gives $c_{l m}=-(2 l-1) C_{l m}^{r} /(l+$ 1) at $\lambda=0$. We conclude that

$$
\begin{align*}
& \lim _{\lambda \rightarrow 0}\left(e^{\lambda(1-r)} \boldsymbol{u}^{H}-\frac{\nabla p}{\lambda^{2}}\right)=\sum_{l=1}^{\infty}\left(-\frac{(l-2) r^{2} \nabla p_{-l-1}}{2 l(2 l-1)}\right. \\
& \left.+\frac{(l+1) \boldsymbol{r} p_{-l-1}}{l(2 l-1)}+\nabla \Phi_{-l-1}\right) \tag{41}
\end{align*}
$$

where we introduced

$$
\begin{equation*}
\Phi_{-l-1} \equiv-\sum_{m=-l}^{m=l} \frac{Y_{l m}(\theta, \phi)}{(l+1) r^{l+1}}\left(\int u_{r} Y_{l m}^{*} d \Omega+\frac{C_{l m}^{r}}{2}\right) \tag{42}
\end{equation*}
$$

which by Eq. (30) reproduces Lamb's solution given by Eq. (29). In fact, the presented formulas provide useful explicit form of the Lamb's solution.

Difficulties in analytic continuation of Lamb's solution-We saw that despite that Eq. (30) generalizes Lamb's solution to a finite frequency, it does so not in the term-by-term way. The reason for this is that finitefrequency is a singular perturbation because at distances larger than the viscous penetration depth, the finitefrequency and zero-frequency solutions are very different however small $\lambda$ is. Other aspect of this difference is that the first term in Eq. (30) has a power-law behavior whereas the remaining terms decay exponentially with $r$ : both $X$ and $\boldsymbol{u}^{H}$ have power-law type dependence on $r$, see Eq. (33) and equivalent form of $\boldsymbol{u}^{H}$ given by

$$
\begin{align*}
& \boldsymbol{u}^{H}=\sum_{l m} C_{l m}^{r}\left(\frac{\hat{\boldsymbol{r}} Y_{l m}}{\lambda \mathcal{P}_{l-1}\left(\lambda^{-1}\right) r^{2}} \mathcal{P}_{l}\left(\frac{1}{\lambda r}\right)\right.  \tag{43}\\
& \left.-\left(\frac{l}{r} \mathcal{P}_{l}\left(\frac{1}{\lambda r}\right)+\lambda \mathcal{P}_{l-1}\left(\frac{1}{\lambda r}\right)\right) \frac{\nabla Y_{l m}}{l(l+1) \lambda \mathcal{P}_{l-1}\left(\lambda^{-1}\right)}\right)
\end{align*}
$$

as can be seen from Eqs. (15)-(16), (19) and (28). The difference in behavior holds because pressure at finite frequency still solves the Laplace equation however the homogeneous solution solves the vector Helmholtz equation. Finally it must be observed that gradients of elementary solutions, $\nabla \chi_{l m}^{\lambda}$, see Eq. (17) solve the vector Helmholtz equation however have finite divergence at non-zero frequency. This is the reason why $\boldsymbol{u}^{H}$ term cannot be constructed as superposition of $\nabla \chi_{l m}^{\lambda}$ terms.

## IV. SMALL FREQUENCY CORRECTIONS TO THE STEADY STOKES LIMIT

In this section we consider the flow at small but finite frequency and describe the corrections to the Stokes limit.

## A. Solution via frequency-independent coefficients

If we consider the solution's dependence on the frequency at fixed flow on the boundary, then the frequency dependence comes both from the expansion coefficients and the functional form of the solution. In many situations, including the low and high frequency limits, it is advantageous to have a representation of the solution written via frequency-independent coefficients $I_{l m}$ defined by

$$
\begin{equation*}
I_{l m}(\boldsymbol{W})=\int_{r=1} \boldsymbol{u} \cdot \boldsymbol{W}_{l m}^{*} d \Omega \tag{44}
\end{equation*}
$$

where $\boldsymbol{W}$ is any of the vector spherical harmonics. We rewrite $\boldsymbol{u}$ in Eq. (19) as

$$
\begin{equation*}
\boldsymbol{u}=\sum_{l m} \boldsymbol{u}_{l m}, \quad \boldsymbol{u}_{l m}=A_{l m}^{Y} \boldsymbol{Y}_{l m}+A_{l m}^{\Psi} \boldsymbol{\Psi}_{l m}+A_{l m}^{\Phi} \boldsymbol{\Phi}_{l m} \tag{45}
\end{equation*}
$$

where we defined functions of radial coordinate

$$
\begin{align*}
& A_{l m}^{Y}=\frac{\left(I_{l m}(\boldsymbol{\Psi})+l I_{l m}(\boldsymbol{Y})\right)\left(A_{l}(1)-r^{l} A_{l}(r)\right)+I_{l m}(\boldsymbol{Y})}{r^{l+2}} \\
& A_{l m}^{\Psi}=\left(I_{l m}(\boldsymbol{\Psi})+l I_{l m}(\boldsymbol{Y})\right)\left(\frac{B_{l-1}(r)}{r l(l+1)}-\frac{A_{l}(1)-r^{l} A_{l}(r)}{r^{l+2}(l+1)}\right) \\
& -\frac{I_{l m}(\boldsymbol{Y})}{r^{l+2}(l+1)}, \quad A_{l m}^{\Phi}=I_{l m}(\boldsymbol{\Phi}) \frac{B_{l}(r)}{r l(l+1)}, \tag{46}
\end{align*}
$$

and we introduced

$$
\begin{equation*}
A_{l}(r)=\frac{\sqrt{r} K_{l+1 / 2}(\lambda r)}{\lambda K_{l-1 / 2}(\lambda)}, B_{l}(r)=\frac{\sqrt{r} K_{l+1 / 2}(\lambda r)}{K_{l+1 / 2}(\lambda)} \tag{47}
\end{equation*}
$$

## B. Small frequency expansion

We study the solution in the limit of low frequency assuming that the velocity field prescribed on the sphere surface has a fixed functional form independent of the frequency. The expansion is singular and its parameter
is $\sqrt{\omega}$ and not $\omega$ as would be the case if Eq. (3) could be solved by a series in $\omega$ (thus summation of infinite number of terms in such a series is done by our solution implicitly). We study the solution up to quadratic order in $\sqrt{\omega}$ by deriving the asymptotic expansion of $\boldsymbol{u}_{l m}$ in Eq. (45)

$$
\begin{equation*}
\boldsymbol{u}_{l m}=\boldsymbol{u}_{l m}^{0}+\lambda \boldsymbol{u}_{l m}^{1}+\lambda^{2} \boldsymbol{u}_{l m}^{2}+\ldots \tag{48}
\end{equation*}
$$

where dots stand for higher order terms. This expansion, by definition, is a quadratic polynomial in $\lambda$. We observe that the force acting on a sphere moving at a given timedependent velocity in the fluid is also quadratic in $\lambda$. Hence interpretation of the terms can be transferred from that in the study of the force [1].

The solution's form given by Eqs. (45)-(46) reduces the study to two functions $A_{l}$ and $B_{l}$ defined after the equations. We have the low-frequency expansions

$$
\begin{align*}
& r^{l} A_{l}(r)-A_{l}(1)=-\frac{r^{2}-1}{2}+\frac{\left(r^{2}-1\right)^{2}}{8(2 l-3)} \lambda^{2} \\
& +\frac{\delta_{l 1}}{6}(r-1)^{2}(1+2 r) \lambda(1+\lambda) \tag{49}
\end{align*}
$$

and

$$
\begin{align*}
& B_{l}(r)=\frac{1}{r^{l}}+\lambda^{2} \frac{\left(1-r^{2}\right)}{2 r^{l}(2 l-1)}, \quad l>0  \tag{50}\\
& B_{l-1}(r)=\frac{1}{r^{l-1}}+\lambda(1+\lambda) \delta_{l 1}(1-r)+\frac{\lambda^{2}\left(1-r^{2}\right)}{2 r^{l-1}(2 l-3)}
\end{align*}
$$

We consider different order terms in $\lambda$.

## C. Leading order: yet another form of general solution of steady Stokes flow

We derived in the previous section Lamb's solution for the steady Stokes flow by taking the limit $\lambda \rightarrow 0$ in our solution. In this subsection we study the same limit for deriving a different representation. We have from Eq. (45) at $\lambda=0$

$$
\begin{equation*}
\boldsymbol{u}_{l m}^{0}=a_{l m}^{Y} \boldsymbol{Y}_{l m}+a_{l m}^{\Psi} \boldsymbol{\Psi}_{l m}+a_{l m}^{\Phi} \mathbf{\Phi}_{l m} \tag{51}
\end{equation*}
$$

where

$$
\begin{align*}
& a_{l m}^{Y}=\frac{\left(I_{l m}(\boldsymbol{\Psi})+l I_{l m}(\boldsymbol{Y})\right)\left(r^{2}-1\right)+2 I_{l m}(\boldsymbol{Y})}{2 r^{l+2}}  \tag{52}\\
& a_{l m}^{\Psi}=\frac{I_{l m}(\boldsymbol{\Psi})+l I_{l m}(\boldsymbol{Y})}{r^{l} l(l+1)}-\frac{a_{l m}^{Y}}{l+1}, \quad a_{l m}^{\Phi}=\frac{I_{l m}(\boldsymbol{\Phi})}{r^{l+1} l(l+1)} .
\end{align*}
$$

Given a boundary condition on the steady Stokes flow the coefficients $I_{l m}$, defined in Eq. (44), can be readily obtained from Eqs. (24)-(25). This gives a general solution of steady Stokes flow which is seemingly missing in the literature. We believe that it is simpler than several representations provided in [1]. For instance the adjoint method [1] represents the solution as expansion in the
surface vector fields $\boldsymbol{A}_{l m}, \boldsymbol{B}_{l m}, \boldsymbol{C}_{l m}$ defined as follows

$$
\begin{align*}
\boldsymbol{A}_{l m} & =l \tilde{Y}_{l m} \hat{r}+\partial_{\theta} \tilde{Y}_{l m} \hat{\theta}+\frac{\partial_{\phi} \tilde{Y}_{l m}}{\sin \theta} \hat{\phi}  \tag{53}\\
\boldsymbol{B}_{l m} & =-(l+1) \tilde{Y}_{l m} \hat{r}+\partial_{\theta} \tilde{Y}_{l m} \hat{\theta}+\frac{\partial_{\phi} \tilde{Y}_{l m}}{\sin \theta} \hat{\phi} \\
\boldsymbol{C}_{l m} & =\frac{\partial_{\phi} \tilde{Y}_{l m}}{\sin \theta} \hat{\theta}-\partial_{\theta} \tilde{Y}_{l m} \hat{\phi} ; \quad \tilde{Y}_{l m}=(-1)^{m} P_{l}^{m}(\cos \theta) e^{i m \phi}
\end{align*}
$$

The above fields are linear combinations of the VSH. We demonstrate how the transition between the expansions can be made. It is easy to obtain

$$
\begin{align*}
& \tilde{Y}_{l m} \hat{r}=\frac{\boldsymbol{A}_{l m}-\boldsymbol{B}_{l m}}{2 l+1}  \tag{54}\\
& \partial_{\theta} \tilde{Y}_{l m} \hat{\theta}+\frac{\partial_{\phi} \tilde{Y}_{l m}}{\sin \theta} \hat{\phi}=\frac{(l+1) \boldsymbol{A}_{l m}+l \boldsymbol{B}_{l m}}{2 l+1}
\end{align*}
$$

We find using the definition in Eq. (17)

$$
Y_{l m}=\eta_{l m} \tilde{Y}_{l m}, \eta_{l m}=(-1)^{m} \sqrt{\frac{(2 l+1)}{4 \pi} \frac{(l-m)!}{(l+m)!}}
$$

This gives by comparing Eqs. (54) and (9) that

$$
\begin{align*}
& \boldsymbol{Y}_{l m}=\frac{\eta_{l m}}{2 l+1}\left(\boldsymbol{A}_{l m}-\boldsymbol{B}_{l m}\right), \boldsymbol{\Phi}_{l m}=\eta_{l m} \boldsymbol{C}_{l m} \\
& \boldsymbol{\Psi}_{l m}=\frac{\eta_{l m}}{2 l+1}\left((l+1) \boldsymbol{A}_{l m}+l \boldsymbol{B}_{l m}\right) \tag{55}
\end{align*}
$$

The inverse transformation is

$$
\begin{equation*}
\boldsymbol{A}_{l m}=\frac{\boldsymbol{\Psi}_{l m}+l \boldsymbol{Y}_{l m}}{\eta_{l m}}, \quad \boldsymbol{B}_{l m}=\frac{\boldsymbol{\Psi}_{l m}-(l+1) \boldsymbol{Y}_{l m}}{\eta_{l m}},( \tag{56}
\end{equation*}
$$

and $\boldsymbol{C}_{l m}=\boldsymbol{\Phi}_{l m} / \eta_{l m}$. The advantage of the representation provided here is that the VSH, which are orthogonal at each point, seem to be simpler than the surface vector fields, which are not. Most significantly the coefficients of the expansion are given by simple Eqs. (24). Similar advantages hold in comparison with other respresentations in [1].

## D. Leading order correction in frequency

Using Eqs. (49)-(51) in Eqs. (45)-(46) it is easy to find that the linear term in $\lambda$ appears only for $l=1$ so that $\boldsymbol{u}_{l m}^{1}=\delta_{l 1} \boldsymbol{u}_{1 m}^{1}$. We have

$$
\begin{align*}
& \boldsymbol{u}_{1 m}^{1}=\frac{(1-r)}{12 r^{3}}\left(I_{1 m}(\boldsymbol{\Psi})+I_{1 m}(\boldsymbol{Y})\right) \\
& \times\left(\left(1+r+4 r^{2}\right) \boldsymbol{\Psi}_{1 m}-2\left(1+r-2 r^{2}\right) \boldsymbol{Y}_{1 m}\right) \tag{57}
\end{align*}
$$

This correction is proportional to square root of the frequency similarly to correction for the fundamental solution of the unsteady Stokes equations [1]. In time domain the correction is non-local and has the structure of the Basset memory integral [1].

## E. Next order, linear correction in frequency

We have using Eqs. (49)-(51) in Eqs. (45)-(46)

$$
\begin{equation*}
\boldsymbol{u}_{l m}^{2}=a_{l m}^{Y} \boldsymbol{Y}_{l m}+a_{l m}^{\Psi} \boldsymbol{\Psi}_{l m}+a_{l m}^{\Phi} \boldsymbol{\Phi}_{l m} \tag{58}
\end{equation*}
$$

where we defined functions of radial coordinate $a(r)$ (whose domain of definition for $a^{\Psi}$ and $a^{\Phi}$ is $l>0$ due to $\boldsymbol{\Psi}_{00}=\boldsymbol{\Phi}_{00}=0$ ) by

$$
\begin{align*}
& a_{l m}^{Y}=-\left(\frac{\left(r^{2}-1\right)^{2}}{8(2 l-3)}+\frac{\delta_{l 1}(r-1)^{2}(1+2 r)}{6}\right)  \tag{59}\\
& \times \frac{\left(I_{l m}(\boldsymbol{\Psi})+l I_{l m}(\boldsymbol{Y})\right)}{r^{l+2}} ; \quad a_{l m}^{\Phi}=\frac{\left(1-r^{2}\right) I_{l m}(\mathbf{\Phi})}{2 r^{l+1} l(l+1)(2 l-1)}, \\
& a_{l m}^{\Psi}=\left(I_{l m}(\boldsymbol{\Psi})+l I_{l m}(\boldsymbol{Y})\right)\left(\frac{\delta_{l 1}(1-r)}{r l(l+1)}+\frac{\left(1-r^{2}\right)}{2 r^{l} l(l+1)(2 l-3)}\right. \\
& \left.+\frac{\left(r^{2}-1\right)^{2}}{8(2 l-3) r^{l+2}(l+1)}+\frac{\delta_{l 1}(r-1)^{2}(1+2 r)}{6 r^{l+2}(l+1)}\right)
\end{align*}
$$

In time domain this correction is local.

## V. FLOW AROUND AN OSCILLATING SPHERE

General oscillatory motion of a rigid sphere can be decomposed into the sum of oscillatory translations and rotations. Both problems were solved previously. Here we demonstrate that these problems give insight into which properties of the boundary flow give rise to various terms in the solution given by Eq. (30).

## A. Periodic translations

The most well-known solution of unsteady Stokes equations (there are not so many) is the flow due to rigid sphere's oscillations in infinite fluid which was solved by Stokes [1]. Here the flow on the sphere is given in frequency domain by a constant complex vector $\boldsymbol{U}$. In this case both the surface divergence and radial component of vorticity at $r=1$ vanish. We find from Eq. (24) that

$$
\begin{align*}
\int \boldsymbol{u} \cdot \boldsymbol{Y}_{l m}^{*} d \Omega & =\int U_{r} Y_{l m}^{*} d \Omega, \quad \int \boldsymbol{u} \cdot \mathbf{\Phi}_{l m}^{*} d \Omega=0 \\
\int \boldsymbol{u} \cdot \boldsymbol{\Psi}_{l m}^{*} d \Omega & =2 \int U_{r} Y_{l m}^{*} d \Omega \tag{60}
\end{align*}
$$

Thus the coefficients reduce to the calculation of

$$
\begin{align*}
& \int U_{r} Y_{l m}^{*} d \Omega=U_{x} \int \sin \theta \cos \phi Y_{l m}^{*} d \Omega \\
& +U_{y} \int \sin \theta \sin \phi Y_{l m}^{*} d \Omega+U_{z} \int \cos \theta Y_{l m}^{*} d \Omega \tag{61}
\end{align*}
$$

The straightforward, yet cumbersome, calculations are provided in Appendix C. It is found that the flow is determined by $l=1$ term of the series solution, i. e., $\boldsymbol{u}_{l m}$ in Eq. (45) is proportional to $\delta_{l 1}$ (similar fact holds for the
counterpart problem for the steady Stokes flow). The solution has the general structure provided in Eq. (30) with

$$
\begin{align*}
& p=\left(1+\lambda+\frac{\lambda^{2}}{3}\right) \frac{3 \boldsymbol{U} \cdot \boldsymbol{r}}{2 r^{3}}, \quad X=0  \tag{62}\\
& \boldsymbol{u}^{H}=\frac{3(1+\lambda r)(\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{2 \lambda^{2} r^{3}}+\frac{3(\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{2 r}
\end{align*}
$$

Thus periodic translations of the sphere generate the flow with zero toroidal component. The agreement with the known solution is confirmed in Appendix C. We provide for the reference below the surface traction which is associated with the oscillations. This is given by the value of $\sigma_{i k} \hat{r}_{k}$ on the particle surface where $\sigma_{i k}$ the stress tensor [1, 30]

$$
\begin{equation*}
\sigma_{i k} \equiv-p \delta_{i k}+\frac{\partial v_{i}}{\partial x_{k}}+\frac{\partial v_{k}}{\partial x_{i}} . \tag{63}
\end{equation*}
$$

It is found by direct calculation that

$$
\begin{equation*}
\sigma \hat{\boldsymbol{r}}=-\frac{3(1+\lambda) \boldsymbol{U}+\lambda^{2}(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{2} \tag{64}
\end{equation*}
$$

In the limit of $\lambda \rightarrow 0$ this reproduces the constant traction of the Stokes flow $-3 \boldsymbol{U} / 2$ whose surface integral gives the Stokes force $-6 \pi \boldsymbol{U}$. Surface integral of the above equation gives the force $\boldsymbol{F}$ on oscillating sphere

$$
\begin{equation*}
\boldsymbol{F}=-6 \pi\left(1+\lambda+\lambda^{2} / 9\right) \boldsymbol{U} \tag{65}
\end{equation*}
$$

cf. [1].

## B. Small frequency expansion

It is of interest to consider the small frequency expansion of the above solution. We have after tedious yet straightforward calculation

$$
\begin{align*}
& \boldsymbol{u}=\frac{3\left(2+2 \lambda+\lambda^{2}\right)(\boldsymbol{U}+(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{8 r}-\lambda(1+\lambda) \boldsymbol{U} \\
& +\frac{(2+\lambda)^{2}(\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{16 r^{3}}+\frac{3 r \lambda^{2}(3 \boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{16}, \tag{66}
\end{align*}
$$

where we discarded terms of order $(\lambda r)^{3}$ and higher. The expansion is in $\lambda r \ll 1$. It is singular with the expansion parameter proportional to the square root of the frequency and not the frequency itself, cf. above. The flow obeys $\boldsymbol{u}(r=1)=\boldsymbol{U}$ in the considered order. The above equation reduces at $\lambda=0$ to the usual steady Stokes flow caused by particle moving at the speed $\boldsymbol{U}$.

The leading order correction in frequency in Eq. (66) contains the constant term $\lambda \boldsymbol{U}$ that does not decay with the distance. Similar fact holds for the Green's function of the unsteady Stokes equations [1]. Remarkably a similar constant correction to the Stokes flow is induced by small but finite stratification [31]. The constant correction demands some care since in the presence of many particles it could produce coherent phenomena. For instance, consider a dilute solution of $N$ particles in a fluid
volume whose linear size is much smaller than $1 / \lambda$ (this situation demands that $1 / \lambda \gtrsim 100$ to allow for large distance between the particles). Since the particles are at distance $r \gg 1$ from each other than in the leading order $i-$ th particle induces uniform flow $-\lambda \boldsymbol{U}_{i}$ at the position of other particles. Therefore if the particles oscillate in the same direction, of which the simplest case is that their oscillatory velocities coincide, $\boldsymbol{U}_{i}=\boldsymbol{U}$, then there is constructive interference of the flows induced by different particles. The total flow induced by other particles at a position of one of the particles may be no longer a small correction to the Stokes flow and there is a possibility for transition to coherent motion of all the particles. The corresponding consistent study is beyond our scope here.

## C. Oscillatory rotation of a sphere

We consider another well-known solution of unsteady Stokes equations, the oscillatory rotation of a sphere [1]. This problem gives an example of a purely toroidal flow.

In the frequency domain the flow on the surface of the unit sphere is $\boldsymbol{\omega} \times \hat{\boldsymbol{r}}$, where $\boldsymbol{\omega}$ is a constant vector of angular velocity. We find readily from Eq. (25), by using $\boldsymbol{\omega} \times \boldsymbol{r}$ as the boundary flow continuation to the whole space, that $\nabla_{s} \cdot \boldsymbol{u}=\nabla \cdot \boldsymbol{u}=0$. The radial component of vorticity obeys at the surface $(\nabla \times \boldsymbol{u})_{r}=2 \omega_{r}$. We conclude from Eqs. (24) that the only non-zero projection is

$$
\begin{align*}
& \int \boldsymbol{u} \cdot \boldsymbol{\Phi}_{l m}^{*} d \Omega=-2 \int_{r=1} Y_{l m}^{*} \omega_{r} d \Omega=-2 \int_{r=1} Y_{l m}^{*} d \Omega \\
& \times\left(\omega_{x} \sin \theta \cos \phi+\omega_{y} \sin \theta \sin \phi+\omega_{z} \cos \theta\right) . \tag{67}
\end{align*}
$$

Calculations brought in Appendix D demonstrate that the solution has the general form given by Eq. (30). Both $p$ and $\boldsymbol{u}^{H}$ vanish and the flow is toroidal

$$
\begin{align*}
& \boldsymbol{u}=\frac{(1+\lambda r) \boldsymbol{\omega} \times \boldsymbol{r}}{(1+\lambda) r^{3}} e^{-\lambda(r-1)}=\nabla \times\left(\boldsymbol{r} e^{\lambda(1-r)} X\right) \\
& X=\frac{(1+\lambda r) \boldsymbol{\omega} \cdot \boldsymbol{r}}{(1+\lambda) r^{3}} e^{-\lambda(r-1)} \tag{68}
\end{align*}
$$

This agrees with the solution provided in [3]. The surface traction is

$$
\begin{equation*}
\sigma \hat{\boldsymbol{r}}=-\boldsymbol{\omega} \times \hat{\boldsymbol{r}}\left(3+\frac{\lambda^{2}}{1+\lambda}\right) \tag{69}
\end{equation*}
$$

see e. g. 32]. The torque is readily found to be [1]

$$
\begin{equation*}
\boldsymbol{T}=-8 \pi \boldsymbol{\omega}-\frac{8 \pi \boldsymbol{\omega} \lambda^{2}}{3(1+\lambda)} \tag{70}
\end{equation*}
$$

The torque's dependence on $\lambda$ in this case is more complex than the parabolic dependence of the force in the problem considered in the previous section.

## VI. FAR FIELD BEHAVIOR OF THE GENERAL SOLUTION

The decomposition $\boldsymbol{u}=\boldsymbol{u}_{s}-\lambda^{-2} \nabla p$ introduced in Eq. (8) is very useful for understanding the long distance behavior of the solutions. We observe that $\boldsymbol{u}_{s}$ decays exponentially far from the sphere, cf. Eq. (16) so that

$$
\begin{equation*}
\boldsymbol{u}=-\frac{\nabla p}{\lambda^{2}}+O(\exp (-\lambda r)), \quad r \gg \delta \tag{71}
\end{equation*}
$$

The formula for $p$ is then provided by Eq. (6) with coefficients in Eq. (26). In non-degenerate case, where at least one of $c_{1 m}$ is non-zero, we have

$$
\begin{equation*}
p=\sum_{m=-1}^{m=1} \frac{c_{1 m} Y_{1 m}(\theta, \phi)}{r^{2}}+o\left(\frac{1}{r}\right) \tag{72}
\end{equation*}
$$

where the coefficients $c_{1 m}$ are given by Eq. (26). We observe by using the definitions of $Y_{1 m}$ that in the leading order the pressure can be written as the solution for sphere oscillating with effective velocity $\boldsymbol{U}_{\text {eff }}$

$$
\begin{equation*}
p=\left(1+\lambda+\frac{\lambda^{2}}{3}\right) \frac{3 \boldsymbol{U}_{e f f} \cdot \boldsymbol{r}}{2 r^{3}} \tag{73}
\end{equation*}
$$

where we introduced

$$
\begin{align*}
& \boldsymbol{U}_{e f f} \equiv \sqrt{\frac{1}{6 \pi}}\left(1+\lambda+\frac{\lambda^{2}}{3}\right)^{-1} \nabla\left(\left(c_{1,-1}-c_{11}\right) x\right. \\
& \left.-i\left(c_{1,-1}+c_{11}\right) y+\sqrt{2} c_{10} z\right) \tag{74}
\end{align*}
$$

cf. Eq. (118). Thus at large distances any source of boundary flow looks as an oscillating rigid sphere. This is true if the above solution does not become trivial for a given boundary flow, in which case higher order terms must be considered leading to rotating sphere and similar terms.

## VII. AXIALLY SYMMETRIC CASE AND THE SOLUTION OF RAO

We consider the reduction of our solution in the case of axially symmetric flows with

$$
\begin{equation*}
\boldsymbol{u}=u(r, \theta) \hat{\boldsymbol{r}}+v(r, \theta) \hat{\boldsymbol{\theta}}, \tag{75}
\end{equation*}
$$

where $u$ and $v$ are the radial and polar components of the flow respectively. This solution applies if the boundary conditions at $r=1$ have the form in the equation above. The coefficients $\tilde{c}_{l m}$ vanish in this case, see the last of Eqs. (24) and Eq. (34). Thus $X$ component in the solution given by Eq. (30) is zero. The coefficients $c_{l m}$ and $\tilde{c}_{l m}^{r}$ vanish if $m \neq 0$. We find projecting the solution given by Eq. (30) onto the radial direction that

$$
\begin{equation*}
u=-\sum_{l=1}^{\infty}\left(\frac{D_{l}(\lambda)}{r^{l+2}}+\frac{F_{l}(\lambda) K_{l+1 / 2}(\lambda r)}{r^{3 / 2}}\right) P_{l}(\cos \theta) \tag{76}
\end{equation*}
$$

where we used $Y_{l 0}=P_{l}(\cos \theta) \sqrt{(2 l+1) /(4 \pi)}$, see Eq. (7). We defined the coefficients

$$
\begin{equation*}
D_{l} \equiv-\sqrt{\frac{2 l+1}{4 \pi}} \frac{(l+1) c_{l 0}}{\lambda^{2}}, \quad F_{l} \equiv-e^{\lambda} \sqrt{\frac{(2 l+1) \lambda}{2}} \frac{c_{l 0}^{r}}{\pi} \tag{77}
\end{equation*}
$$

The explicit form of the coefficients is found by using the formulas for $c_{l 0}$ and $\tilde{c}_{l 0}^{r}$. Calculations brought in Appendix $F$ give

$$
\begin{align*}
& D_{l}=-\frac{(2 l+1)\left(l \mathcal{P}_{l}\left(\lambda^{-1}\right)+\lambda \mathcal{P}_{l-1}\left(\lambda^{-1}\right)\right)}{2 \lambda \mathcal{P}_{l-1}\left(\lambda^{-1}\right)} \int_{-1}^{1} P_{l}(x) u(x) d x \\
& -\frac{(2 l+1) \mathcal{P}_{l}\left(\lambda^{-1}\right)}{2 \lambda \mathcal{P}_{l-1}\left(\lambda^{-1}\right)} \int_{-1}^{1} P_{l}^{1}(x) v(x) d x \tag{78}
\end{align*}
$$

The corresponding formula for $F_{l}$ is readily inferred from the relation between $c_{l m}$ and $\tilde{c}_{l m}^{r}$ given by Eq. (23). The obtained solution reproduces the known solution of 19], see details in the Appendix.

## VIII. IDEAL FLOW AND HIGH FREQUENCY EXPANSION

In this section we demonstrate that the ideal flow approximation, that is often postulated rather than derived, can be obtained from our solution, including the corrections in small but finite viscosity. The limit of small viscosity corresponds to the large $|\lambda|$ limit, see the definition in Sec. III In this limit $\delta$ becomes vanishingly small. The flow outside a narrow boundary layer of width $\delta$ around the sphere the flow is potential up to exponentially small corrections in $|\lambda|$, see Eq. (30). In contrast, the asymptotic expansion of the potential, which is given by $-p / \lambda^{2}$, is a power series in $|\lambda|^{-1}$. Hence, neglecting exponentially small corrections, the expansion with respect to small viscosity or large $|\lambda|$ is obtained by expansion of the potential which is determined by the pressure coefficients $c_{l m}$, see Eq. (6). We have (cf. Eq. (71))

$$
\begin{equation*}
\boldsymbol{u}=\nabla \psi, \quad \psi=\psi_{0}+\frac{\psi_{1}}{\lambda}+\frac{\psi_{2}}{\lambda^{2}}+\ldots \tag{79}
\end{equation*}
$$

In order to derive the asymptotic expansion of $c_{l m}$ we rewrite Eq. (26) as

$$
\begin{equation*}
c_{l m}=\frac{\lambda^{2} b_{l m}}{l+1}+\frac{\lambda \mathcal{P}_{l}\left(\lambda^{-1}\right)\left((l+2) b_{l m}-d_{l m}\right)}{(l+1) \mathcal{P}_{l-1}\left(\lambda^{-1}\right)} \tag{80}
\end{equation*}
$$

where we introduced the coefficients

$$
\begin{equation*}
b_{l m} \equiv \int_{r=1} Y_{l m}^{*} u_{r} d \Omega, \quad d_{l m} \equiv \int Y_{l m}^{*} \nabla_{s} \cdot \boldsymbol{u} d \Omega \tag{81}
\end{equation*}
$$

We will assume in the asymptotic expansions below that $b_{l m}$ and $d_{l m}$ are fixed i.e. the flow on the boundary sphere is prescribed and we study how the flow that it generates depends on the viscosity. We have from Eq. (16) that

$$
\begin{equation*}
\frac{\mathcal{P}_{l}\left(\lambda^{-1}\right)}{\mathcal{P}_{l-1}\left(\lambda^{-1}\right)}=1+\frac{l}{\lambda}+O\left(\frac{1}{\lambda^{2}}\right) . \tag{82}
\end{equation*}
$$

## A. Ideal flow approximation

We find from the above that

$$
\begin{equation*}
c_{l m}=\frac{\lambda^{2} b_{l m}}{l+1}+O(|\lambda|) \tag{83}
\end{equation*}
$$

We conclude that in the leading order in small viscosity we have

$$
\begin{equation*}
\boldsymbol{u}=\nabla \psi_{0}, \quad \psi_{0}=-\sum_{l m} \frac{b_{l m} Y_{l m}(\theta, \phi)}{(l+1) r^{l+1}} \tag{84}
\end{equation*}
$$

Thus, in this limit the solution is fully determined by the normal component of the flow $u_{r}$. In fact, we have

$$
\begin{equation*}
\partial_{r} \psi_{0}(r=1)=\sum_{l m} Y_{l m}(\theta, \phi) \int_{r=1} Y_{l m}^{*} u_{r} d \Omega \tag{85}
\end{equation*}
$$

which demonstrates that the normal component of the solution provided by Eq. (79) coincides with $u_{r}$ on the sphere surface. We have recovered the prescription of the ideal flow [3].

## B. Corrections to ideal flow approximation

We consider small viscosity corrections to the ideal flow approximation, which seemingly were not provided previously. We have from the above that

$$
\begin{equation*}
c_{l m}=\frac{\lambda^{2} b_{l m}}{l+1}+\frac{(\lambda+l)\left((l+2) b_{l m}-d_{l m}\right)}{l+1}+O\left(\frac{1}{|\lambda|}\right) . \tag{86}
\end{equation*}
$$

We find that the corrections to the ideal flow $\psi_{1}$ and $\psi_{2}$ in Eq. (79) are given by

$$
\begin{align*}
& \psi_{1}=\sum_{l m} \frac{\left(d_{l m}-(l+2) b_{l m}\right) Y_{l m}(\theta, \phi)}{(l+1) r^{l+1}}  \tag{87}\\
& \psi_{2}=\sum_{l m} \frac{l\left(d_{l m}-(l+2) b_{l m}\right) Y_{l m}(\theta, \phi)}{(l+1) r^{l+1}}=-\partial_{r}\left(r \psi_{1}\right) .
\end{align*}
$$

Several things are to be observed about the correction. It is proportional to square root of the viscosity and not the viscosity itself. It involves tangential components of the surface flow via the $d_{l m}$ coefficients. Finally, in contrast to the ideal flow approximation, the leading order correction is not local in time. As can be seen by inverse Fourier transform the flow in this order has memory and cannot be written in terms of the instantaneous flow on the surface (here we refer to arbitrary, not necessarily periodic, time-dependent flow on the surface).

## C. Force

The ideal flow approximation gives that the force on the sphere is determined by pressure component of the
stress tensor only and is given by

$$
\begin{align*}
& \boldsymbol{F}_{i d}=-\int_{r=1} p \hat{\boldsymbol{r}} d \Omega=-\sum_{l m} c_{l m} \int_{r=1} Y_{l m}(\theta, \phi) \hat{\boldsymbol{r}} d \Omega \\
& =-\sum_{m} c_{1 m} \sqrt{\frac{2 \pi}{3}}\left(\hat{x}\left(\delta_{m,-1}-\delta_{m, 1}\right)-i \hat{y}\left(\delta_{m,-1}+\delta_{m, 1}\right)\right. \\
& \left.+\hat{z} \sqrt{2} \delta_{m 0}\right) \tag{88}
\end{align*}
$$

where we used Eq. (146) from Appendix E] The same Appendix shows that generally the force at any $\lambda$ obeys the parabolic dependence that can be written as

$$
\begin{align*}
& \boldsymbol{F}=-\lambda^{2} \sqrt{\frac{\pi}{6}}\left(\hat{\boldsymbol{x}}\left(b_{1,-1}-b_{11}\right)-i \hat{\boldsymbol{y}}\left(b_{1,-1}+b_{11}\right)+\hat{\boldsymbol{z}} \sqrt{2} b_{10}\right) \\
& +(1+\lambda) \sqrt{\frac{3 \pi}{2}}\left(\hat{\boldsymbol{x}}\left(d_{1,-1}-3 b_{1,-1}-d_{11}+3 b_{11}\right)\right. \\
& \left.-i \hat{\boldsymbol{y}}\left(d_{1,-1}-3 b_{1,-1}+d_{11}-3 b_{11}\right)+\hat{\boldsymbol{z}} \sqrt{2}\left(d_{10}-3 b_{10}\right)\right) . \tag{89}
\end{align*}
$$

Comparison of the last two equations, using Eq. (83), demonstrates that in the leading, quadratic, order, $\boldsymbol{F}$ and $\boldsymbol{F}_{i d}$ coincide, confirming the ideal flow prescription. In higher orders $\boldsymbol{F}$ has also the viscous stress contributions, see Appendix E

## IX. GENERAL SOLUTION OF BRINKMAN EQUATIONS

The analysis proposed in this paper can be applied directly to the study of the Brinkman equations

$$
\begin{equation*}
\frac{\nu \boldsymbol{v}}{k}+\rho^{-1} \nabla p=\nu \nabla^{2} \boldsymbol{v} \tag{90}
\end{equation*}
$$

These equations describe the flow in the porous medium at low volume fraction of solids [27, 28] and can serve as the starting point for the derivation of Darcy's law. Here $k$ is the damping coefficient and $\nu / k$ is the permeability [27], see [33] for recent discussion of the coefficients and more references.

We observe that Eq. (19) provides the solution of Eqs. (5) also when $\lambda$ is a complex number whose square is not purely imaginary. In the case of $\omega=i \nu / k$, where $\lambda^{2}=a^{2} / k$, Eq. (5) becomes the Brinkman equations. Thus Eq. (19) with $\lambda=a / \sqrt{k}$ is the general solution of these equations and all the rest of our considerations can be transferred by analytic continuation. At large distances we find the well-known Darcy's law $\boldsymbol{v}=-\left(k / a^{2}\right) \nabla p$.

In fact, the general solution presented here can be used to study the more general equations

$$
\begin{equation*}
\left(-i \omega+\frac{\nu}{k}\right) \boldsymbol{v}+\rho^{-1} \nabla p=\nu \nabla^{2} \boldsymbol{v} \tag{91}
\end{equation*}
$$

that interpolate between the unsteady Stokes and Brinkman equations. The above equations, which are called the unsteady Brinkman equations, can be considered as a model of unsteady flow in the porous medium
[34]. It is plausible that they can actually be derived from unsteady Stokes equations in the porous medium similarly to the derivation of the usual Brinkman equations from time-independent Stokes equations, see references in [28]. However, this derivation is beyond our scope here. Eqs. (91) can also be studied by analytic continuation of our solution to a complex frequency which is neither real nor imaginary.

We remark that a complete solution of the timeindependent Brinkman equations was proposed previously in [35] and its generalization for the unsteady case in [34]. The properties of the solutions are not readily available from the proposed representations. Thus the reduction from the Brinkman equations to the Darcy's law is not as immediate as in solutions proposed here.

## X. CONCLUSIONS AND FUTURE WORK

We presented the general solution of unsteady Stokes equations as a series in vector spherical harmonics (VSH). The solution allows to fix the flow in the whole space given the values of the flow at a spherical boundary. The solution has a Lamb-type form and is given by Eq. (30) which is a main result of this paper.

The solution's form as a superposition of a partial solution and a solenoidal solution of a vector Helmoholtz equation provides unique insights into the general behavior of the solutions. Thus it is immediate that at distances from the sphere that are much larger than the viscous penetration depth a generalized Darcy's law holds, $\boldsymbol{u}=-i \nabla p /(\rho \omega)$. The pressure is a series of spherical harmonics whose coefficients are determined by projections of the surface radial velocity and divergence on the spherical harmonics. In the leading order in the distance the far flow is that due to an oscillating rigid sphere with an effective amplitude of the oscillations.

Both limits of large and small frequency are singular. The solutions in these limits is given by an asymptotic series in $1 / \sqrt{R o}$ and $\sqrt{R o}$, respectively. Here $R o=a^{2} \omega / \nu$ so the limits can also be interpreted as those of small and large viscosity, respectively.

The leading order term in the small frequency expansion is Lamb's solution of the steady Stokes equations. We also provide a different form of the solution of the steady Stokes equations, similar to that in the adjoint method [1]. The next order correction, proportional to $\sqrt{R o}$ is peculiar - it consists of the VSH with $l=1$ only. It is not local in the time domain so the leading order correction to the steady Stokes equations limit is nonlocal. The next order, proportional to $R o$ and linear in the frequency, is local in time.

The limit of $R o \rightarrow \infty$ can be considered as that of viscosity going to zero. We prove that in the leading order the flow is potential and can be derived from the usual ideal flow prescription (this prescription is usually
unproved) [3, 25]. The next order corrections in $1 / \sqrt{R o}$ are also potential. We provide explicit formulas for the potentials of orders $1 / \sqrt{R o}$ and $1 / R o$.

The power of the general solution, besides giving qualitative insights above, is that it can be used for solving uniformly for unsteady Stokes flow caused by any boundary condition. We provide some cases where the solution can be applied.

A wide-spread situation is the case of a rigid particle, considered as a sphere, in an external unsteady Stokes flow e.g. a shear flow. Representing the solution as the sum of the external flow and the perturbation flow due to the particle, we find that perturbation flow obeys equations to which the solution presented here can be applied. Thus the boundary flow is a sum of the external flow and superposition of rigid translation and rotation. Further conditions are implied by the mechanics. For instance if the inertia of the particle is negligible then the conditions of zero force and torque on the particle must be imposed. Using the formulas for the force and the torque presented in Appendix E the general solution can be derived readily.

Similar calculational scheme arises for the problem of spherical squirmers, which is one of the most popular model for swimming at low Reynolds number 36]. The particle self propels by periodically changing its shape that remains all the time close to a sphere. The flow is then determined by the boundary conditions on the moving envelope which in the leading order reduces to periodic flow on the spherical boundary. The flow then can be obtained similarly to the above case of rigid particle in external flow.

We can consider similarly the problem of unsteady motion of a slightly deformed sphere. Perturbation theory in small deviations from sphericity leads to the problem of unsteady Stokes equations with given boundary conditions on the sphere, similarly to the steady Stokes counterpart problem [2, 37]. Our solution can then be used for deriving the terms of the perturbation series.

Other type of problems where our general solution can be useful is many body problems. An example is a solution of rigid spheres driven by external periodic flow. Assuming that the solution is dilute, the total flow perturbation due to the particles is given by superposition of perturbations of each particle. We demonstrated that constructive interference between the single-particle perturbations is possible. Thus, collective phenomena can be envisaged whose future study is intriguing. Our solution is also a good starting point for the corresponding numerical algorithm design. The solution can be used similarly to how in the case of steady Stokes equations Lamb's solution is used as a starting point for simulating clusters of particles [18].

The above directions demonstrate that there a lot of possibilities for future applications of the solution presented here. Thus we believe that our work provides a significant contribution to the fluid mechanics.
[1] S. Kim and S. J. Karrila, Microhydrodynamics: principles and selected applications, (Courier Corporation, 2013).
[2] J. Happel and H. Brenner, Low Reynolds number hydrodynamics, (Kluwer, Boston, 1983).
[3] L. D. Landau and E. M. Lifshitz, Fluid Mechanics, 3rd ed. (Pergamon Press, Oxford, 1976).
[4] H. Lamb, Hydrodynamics, (University Press, 1924).
[5] R. Schmitz and B. U. Felderhof, Creeping flow about a spherical particle, Phys. A 113, 90 (1982).
[6] R. G. Barrera, G. A. Estevez, and J. Giraldo, Vector spherical harmonics and their application to magnetostatics, Eur. J. Phys., 6, 287 (1985).
[7] P. F. Papkovich, The representation of the general integral of the fundamental equations of elasticity theory in terms of harmonic functions (in Russian), Izv. Akad. Nauk. SSSR Ser. Mat. 10, 1425 (1932); C. R. Acad. Sci. Paris Ser. A-B 95, 513 (1932).
[8] H. V. Neuber, Ein neuer ansatz zur lösung räumlicher probleme der elastizitätstheorie. der hohlkegel unter einzellast als beispiel, ZAMM-Zeit. Angew. Math. Mech. 14, 203 (1934).
[9] T. Tran-Cong and J. R. Blake, General solutions of the Stokes' flow equations, J. Math. An. App. 90, 72 (1982).
[10] P. M. Naghdi and C. S. Hsu, On a representation of displacements in linear elasticity in terms of three stress functions, J. Math. Mech. 10, 233 (1961).
[11] X. Xinsheng and W. Minzhong, General complete solutions of the equations of spatial and axisymmetric Stokes flow, Quart. J. Mech. App. Math. 44, 537 (1991).
[12] B. S. Padmaathi, G. R. Sekhar, and T. Amaranath, A note on complete general solutions of Stokes equations, Quart. J. Mech. App. Math. 51, 383 (1998).
[13] J. Boussinesq, Equilibre d'elasticite d'un solide sans pesanteur, homogene et isotrope, dont les parties profondes sont maintenues fixes, pendant que sa surface eprouve des pressions ou des deplacements connus, s'annullant hors d'une region restreinte ou ils sont arbitraires, CR Acad Sci 106, 1119 (1888).
[14] D. Palaniappan, S. D. Nigam, T. Amaranath, and R. Usha, Lamb's solution of Stokes's equations: a sphere theorem, Quart. J. Mech. App. Math. 45, 4 (1992).
[15] S. Chandrasekhar, Hydrodynamic and hydromagnetic stability, (Courier Corporation, 2013).
[16] P. Chadwick and E. A. Trowbridge, Elastic wave fields generated by scalar wave functions, Math. Proc. Cam. Phil. Soc. 63, 1177 (1967).
[17] K. B. Ranger, The Stokes drag for asymmetric flow past a spherical cap, ZAMP-Zeit. Angew. Math. Phys 24, 801 (1973).
[18] A. V. Filippov, Drag and torque on clusters of N arbitrary spheres at low Reynolds number, J. Col. Int. Sc. 229, 184 (2000).
[19] R. M. Rao, Mathematical model for unsteady ciliary propulsion, Math. Comput. Model. 10, 839 (1988).
[20] A. Venkatalaxmi, B. S. Padmavathi, and T. Amaranath, A general solution of unsteady Stokes equations, Fluid Dyn. Res. 35, 229 (2004).
[21] C. F. Bohren and D. R. Huffman, Absorption and scattering of light by small particles, (John Wiley and Sons, 2008).
[22] J. A. Stratton, Electromagnetic theory (Vol. 33), (John

Wiley and Sons, 2007).
[23] I. Fouxon, J. Feinberg, and M. Mond, Linear and nonlinear hydromagnetic stability in laminar and turbulent flows, Phys. Rev. E 103, 043104 (2021).
[24] I. Fouxon and A. Leshansky, Fundamental solution of unsteady Stokes equations and force on an oscillating sphere near a wall, Phys. Rev. E 98, 063108 (2018).
[25] G. K. Batchelor, An introduction to fluid dynamics, (Cambridge University Press, 2000).
[26] A. Simha, J. Mo, and P. J. Morrison, Unsteady Stokes flow near boundaries: the point-particle approximation and the method of reflections, J. Fluid Mech. 841, 883 (2018).
[27] H. C. Brinkman, A calculation of the viscous force exerted by a flowing fluid on a dense swarm of particles, Appl. Sci. Res. A1, 27 (1947).
[28] L. Durlofsky and J. F. Brady, Analysis of the Brinkman equation as a model for flow in porous media, Phys. Fluids 30, 3329 (1987).
[29] P. Natalini and P. E. Ricci, Bell polynomials and modified Bessel functions of half-integral order, App. Math. Comp. 268, 270 (2015).
[30] M. R. Maxey and J. J. Riley, Equation of motion for a small rigid sphere in a nonuniform flow, Phys. Fluids 26, 883 (1983).
[31] I. Fouxon and A. Leshansky, Convective stability of turbulent Boussinesq flow in the dissipative range and flow around small particles, Phys. Rev. E 90, 053002 (2014).
[32] I. Fouxon and Y. Or, Inertial self-propulsion of spherical microswimmers by rotation-translation coupling, Phys. Rev. Fluids 4, 023101 (2019).
[33] S. K. Zaripov, R. F. Mardanov, and V. F. Sharafutdinov, Determination of Brinkman Model Parameters Using Stokes Flow Model, Tran. Por. Media 130, 529 (2019).
[34] S. K. Tumuluri and T. Amaranath, A complete general solution of the unsteady Brinkman equations, J. Math. An. App. 461, 1365 (2018).
[35] G. R. Sekhar, B. S. Padmavathi, and T. Amaranath, Complete general solution of the Brinkman equations, ZAMM-Zeit. Angew. Math. Mech. 77, 555 (1997).
[36] M. J. Lighthill, On the squirming motion of nearly spherical deformable bodies through liquids at very small Reynolds numbers, Comm. Pure App. Math. 5, 109 (1952).
[37] H. Brenner, The Stokes resistance of a slightly deformed sphere, Chem. Eng. Sc. 19, 519 (1964).
[38] B. Carrascal, G. A. Estevez, P. Lee, and V. Lorenzo, Vector spherical harmonics and their application to classical electrodynamics, Eur. J. Phys. 12, 184 (1991).
[39] H. Feshbach and P. M. Morse, Methods of theoretical physics, (McGraw-Hill Book Company, inc., 1953).
[40] J. M. Burgers, Second report on viscosity and plasticity, 113, (Nordemann, New York, 1938).

## Appendix A SOLUTION OF VECTOR HELMHOLTZ EQUATION

In this Appendix we provide details of derivation of solution of vector Helmholtz equation for the subsection IIC. The curl of Eq. (13) reads [6],

$$
\begin{align*}
& \nabla \times \boldsymbol{u}_{s}=\sum_{l=1}^{\infty} \sum_{m=-l}^{l}\left(-\frac{l(l+1) c_{l m}^{(2)} \boldsymbol{Y}_{l m}}{r}\right.  \tag{92}\\
& \left.-\left(\frac{d c_{l m}^{(2)}}{d r}+\frac{c_{l m}^{(2)}}{r}\right) \boldsymbol{\Psi}_{l m}+\left(\frac{d c_{l m}^{1}}{d r}+\frac{c_{l m}^{(1)}}{r}-\frac{c_{l m}^{r}}{r}\right) \boldsymbol{\Phi}_{l m}\right)
\end{align*}
$$

Taking curl of the curl and using that incompressibility implies $\nabla^{2} \boldsymbol{u}_{s}=-\nabla \times\left(\nabla \times \boldsymbol{u}_{s}\right)$ we find,

$$
\begin{align*}
& \nabla^{2} \boldsymbol{u}_{s}=\sum_{l=1}^{\infty} \sum_{m=-l}^{l}\left(\frac{l(l+1) \boldsymbol{Y}_{l m}}{r}\left(\frac{d c_{l m}^{(1)}}{d r}+\frac{c_{l m}^{(1)}}{r}-\frac{c_{l m}^{r}}{r}\right)\right. \\
& +\boldsymbol{\Psi}_{l m}\left(\frac{d}{d r}+\frac{1}{r}\right)\left(\frac{d c_{l m}^{(1)}}{d r}+\frac{c_{l m}^{(1)}}{r}-\frac{c_{l m}^{r}}{r}\right)  \tag{93}\\
& \left.-\boldsymbol{\Phi}_{l m}\left[\frac{l(l+1) c_{l m}^{(2)}}{r^{2}}-\left(\frac{d}{d r}+\frac{1}{r}\right)\left(\frac{d c_{l m}^{(2)}}{d r}+\frac{c_{l m}^{(2)}}{r}\right)\right]\right)
\end{align*}
$$

The coefficients obey by $\lambda^{2} \boldsymbol{u}_{s}=\nabla^{2} \boldsymbol{u}_{s}$ that

$$
\begin{align*}
& \lambda^{2} c_{l m}^{r}-\frac{l(l+1)}{r}\left(\frac{d c_{l m}^{(1)}}{d r}+\frac{c_{l m}^{(1)}}{r}-\frac{c_{l m}^{r}}{r}\right)=0,  \tag{94}\\
& \lambda^{2} c_{l m}^{(1)}-\left(\frac{d}{d r}+\frac{1}{r}\right)\left(\frac{d c_{l m}^{(1)}}{d r}+\frac{c_{l m}^{(1)}}{r}-\frac{c_{l m}^{r}}{r}\right)=0, \\
& \lambda^{2} c_{l m}^{(2)}+\frac{l(l+1) c_{l m}^{(2)}}{r^{2}}-\left(\frac{d}{d r}+\frac{1}{r}\right)\left(\frac{d c_{l m}^{(2)}}{d r}+\frac{c_{l m}^{(2)}}{r}\right)=0 .
\end{align*}
$$

The last equation can be written,

$$
\begin{equation*}
\frac{d^{2}}{d r^{2}}\left(r c_{l m}^{(2)}\right)-\left(\lambda^{2}+\frac{l(l+1)}{r^{2}}\right) r c_{l m}^{(2)}=0 \tag{95}
\end{equation*}
$$

The solution that decays at infinity $(R e \lambda>0)$ is,

$$
\begin{equation*}
c_{l m}^{(2)}=\frac{\tilde{c}_{l m} K_{l+1 / 2}(\lambda r)}{\sqrt{r}} \tag{96}
\end{equation*}
$$

where $\tilde{c}_{l m}$ is a constant and we the modified Bessel function $K_{l+1 / 2}$ is defined in Eqs. (15)-(16). The incompressibility condition does not impose any restrictions on $\tilde{c}_{l m}^{(2)}$. We have,

$$
\begin{equation*}
\nabla \cdot \boldsymbol{u}=\sum_{l=1}^{\infty} \sum_{m=-l}^{l}\left(\frac{d c_{l m}^{r}}{d r}+\frac{2 c_{l m}^{r}}{r}-\frac{l(l+1) c_{l m}^{(1)}}{r}\right) Y_{l m} \tag{97}
\end{equation*}
$$

The rest of Eqs. (94) give,

$$
\begin{align*}
& \frac{d\left(r c_{l m}^{(1)}\right)}{d r}-c_{l m}^{r}=\frac{\lambda^{2} r^{2} c_{l m}^{r}}{l(l+1)} \\
& \frac{d^{2}}{d r^{2}}\left(r c_{l m}^{(1)}\right)-\frac{d c_{l m}^{r}}{d r}=\lambda^{2} r c_{l m}^{(1)} \tag{98}
\end{align*}
$$

Consistency of these equations demands that,

$$
\begin{equation*}
\frac{d\left(r^{2} c_{l m}^{r}\right)}{d r}=l(l+1) r c_{l m}^{(1)} \tag{99}
\end{equation*}
$$

which is equivalent to incompressibility condition, see Eq. (97). Using this condition we find coupled equations,

$$
\begin{align*}
& \frac{d\left(r c_{l m}^{(1)}\right)}{d r}=c_{l m}^{r}+\frac{\lambda^{2} r^{2} c_{l m}^{r}}{l(l+1)} \\
& \frac{d\left(r^{2} c_{l m}^{r}\right)}{d r}=l(l+1) r c_{l m}^{(1)} \tag{100}
\end{align*}
$$

Taking derivative of the last equation and using the first,

$$
\begin{equation*}
\frac{d^{2} y}{d r^{2}}-\left[\lambda^{2}+\frac{l(l+1)}{r^{2}}\right] y=0 \tag{101}
\end{equation*}
$$

where $y=r^{2} c_{l m}^{r}$. The solution that vanishes at infinity is,

$$
\begin{equation*}
c_{l m}^{r}=\frac{\tilde{c}_{l m}^{r} K_{l+1 / 2}(\lambda r)}{r^{3 / 2}} \tag{102}
\end{equation*}
$$

where $\tilde{c}_{l m}^{r}$ is a constant. We find using Eq. (99),

$$
\begin{equation*}
c_{l m}^{(1)}=\frac{\tilde{c}_{l m}^{r}}{l(l+1) r} \frac{d\left(r^{1 / 2} K_{l+1 / 2}(\lambda r)\right)}{d r} . \tag{103}
\end{equation*}
$$

We conclude that general solution of Eqs. (5) has the form,

$$
\begin{align*}
& \boldsymbol{u}=\sum_{l=1}^{\infty} \sum_{m=-l}^{l}\left(\left(\frac{\tilde{c}_{l m}^{r} K_{l+1 / 2}(\lambda r)}{r^{3 / 2}}+\frac{(l+1) c_{l m}}{\lambda^{2} r^{l+2}}\right) \boldsymbol{Y}_{l m}\right. \\
& +\left(\frac{\tilde{c}_{l m}^{r}}{l(l+1) r} \frac{d\left(r^{1 / 2} K_{l+1 / 2}(\lambda r)\right)}{d r}-\frac{c_{l m}}{\lambda^{2} r^{l+2}}\right) \boldsymbol{\Psi}_{l m} \\
& \left.+\frac{\tilde{c}_{l m} K_{l+1 / 2}(\lambda r)}{\sqrt{r}} \boldsymbol{\Phi}_{l m}\right) \tag{104}
\end{align*}
$$

where we used Eqs. (8), (12). We can rewrite the above by using

$$
\frac{d K_{\nu}(z)}{d z}=-\frac{\nu K_{\nu}(z)}{z}-K_{\nu-1}(z)
$$

which gives

$$
\left(r^{1 / 2} K_{l+1 / 2}(\lambda r)\right)^{\prime}=-\frac{l K_{l+1 / 2}(\lambda r)+\lambda r K_{l-1 / 2}(\lambda r)}{\sqrt{r}}
$$

The usage of this identity in Eq. (104) gives Eq. (19).

## Appendix B TRANSFORMATION OF THE COEFFICIENTS

In this Appendix details for the identities that underly the transformation of the coefficients of expansion in subsection IID cf. Appendix of 38]. We introduce the field $\tilde{\boldsymbol{u}}$ as the field which has zero radial component and has $r$-independent azimuthal and polar components. It is
set that $\tilde{\boldsymbol{u}}$ coincides with $\boldsymbol{u}$ on the sphere $r=1$. We have then

$$
\begin{align*}
& \int \boldsymbol{u} \cdot \mathbf{\Psi}_{l m}^{*} d \Omega=\int \tilde{\boldsymbol{u}} \cdot \mathbf{\Psi}_{l m}^{*} d \Omega=\int_{r=1} \tilde{\boldsymbol{u}} \cdot \nabla Y_{l m}^{*} d \Omega \\
& =\int_{r=1} \nabla \cdot\left(\tilde{\boldsymbol{u}} Y_{l m}^{*}\right) d \Omega-\int_{r=1} Y_{l m}^{*} \nabla \cdot \tilde{\boldsymbol{u}} d \Omega \tag{105}
\end{align*}
$$

We use that by divergence theorem

$$
\begin{align*}
& 0=\int_{x<1} \nabla \cdot\left(\tilde{\boldsymbol{u}} Y_{l m}^{*}\right) d V=\int_{0}^{1} r d r\left(\nabla \cdot\left(\tilde{\boldsymbol{u}} Y_{l m}^{*}\right)\right)_{r=1} d \Omega \\
& =\frac{1}{2} \int_{r=1} \nabla \cdot\left(\tilde{\boldsymbol{u}} Y_{l m}^{*}\right) d \Omega \tag{106}
\end{align*}
$$

We conclude from the above that

$$
\begin{equation*}
\int \boldsymbol{u} \cdot \boldsymbol{\Psi}_{l m}^{*} d \Omega=-\int Y_{l m}^{*}\left(\frac{\partial\left(u_{\theta} \sin \theta\right)}{\partial \theta}+\frac{\partial u_{\phi}}{\partial \phi}\right) d \theta d \phi,( \tag{107}
\end{equation*}
$$

where we used $\nabla \cdot \tilde{\boldsymbol{u}}$ in spherical coordinates. We develop a similar formula for

$$
\begin{align*}
& \int \boldsymbol{u} \cdot \boldsymbol{\Phi}_{l m}^{*} d \Omega=-\int_{r=1} \tilde{\boldsymbol{u}} \cdot \nabla \times\left(\boldsymbol{r} Y_{l m}^{*}\right) d \Omega \\
& =\int_{r=1} \nabla \cdot\left(\tilde{\boldsymbol{u}} \times\left(\boldsymbol{r} Y_{l m}^{*}\right)\right) d \Omega-\int_{r=1} Y_{l m}^{*} \boldsymbol{r} \cdot \nabla \times \tilde{\boldsymbol{u}} d \Omega,(1 \tag{108}
\end{align*}
$$

where we observed that radial component of the vector field $\tilde{\boldsymbol{u}} \times\left(\boldsymbol{r} Y_{l m}\right)$ is zero and other components depend on $r$ linearly. We have similarly to the above

$$
\begin{align*}
& 0=\int_{x<1} \nabla \cdot\left(\tilde{\boldsymbol{u}} \times\left(\boldsymbol{r} Y_{l m}^{*}\right)\right) d V=\int_{0}^{1} r^{2} d r d \Omega  \tag{109}\\
& \times\left(\nabla \cdot\left(\tilde{\boldsymbol{u}} \times\left(\boldsymbol{r} Y_{l m}^{*}\right)\right)\right)_{r=1}=\frac{1}{3} \int_{r=1} \nabla \cdot\left(\tilde{\boldsymbol{u}} \times\left(\boldsymbol{r} Y_{l m}^{*}\right)\right) d \Omega
\end{align*}
$$

We conclude that

$$
\begin{equation*}
\int \boldsymbol{u} \cdot \boldsymbol{\Phi}_{l m}^{*} d \Omega=\int_{r=1} Y_{l m}^{*}\left(\frac{\partial u_{\theta}}{\partial \phi}-\frac{\partial\left(u_{\phi} \sin \theta\right)}{\partial \theta}\right) d \theta d \phi \tag{110}
\end{equation*}
$$

where we used curl in spherical coordinates. The above gives Eq. (24).

## Appendix C OSCILLATING SPHERE PROBLEM

Here we provide the calculations that reproduce the known solution for the flow around an oscillating sphere used in Sec. $\mathbf{V}$ We observe that the first two terms in the VSH expansion of the solution are non-zero only if $m=1$ or $m=-1$. We use

$$
\begin{equation*}
Y_{l 1}=\sqrt{\frac{(2 l+1)}{4 \pi l(l+1)}} P_{l}^{1}(\cos \theta) \exp (i \phi)=-Y_{l,-1}^{*} \tag{111}
\end{equation*}
$$

and that definition of $P_{l}^{m}$ implies

$$
\begin{align*}
& \int \sqrt{1-x^{2}} P_{l}^{1}(x) d x=-\int\left(1-x^{2}\right) P_{l}^{\prime}(x) d x \\
& =-2 \int x P_{l}(x) d x=-\frac{4 \delta_{l 1}}{3} \tag{112}
\end{align*}
$$

This gives

$$
\begin{align*}
& \int \sin \theta \cos \phi Y_{l m} d \Omega=\sqrt{\frac{2 \pi}{3}} \delta_{l 1}\left(\delta_{m,-1}-\delta_{m, 1}\right) \\
& \int \sin \theta \sin \phi Y_{l m} d \Omega=-i \sqrt{\frac{2 \pi}{3}} \delta_{l 1}\left(\delta_{m,-1}+\delta_{m, 1}\right)( \tag{113}
\end{align*}
$$

We also have $\int \cos \theta Y_{l m}^{*} d \Omega=2 \delta_{l 1} \delta_{m 0} \sqrt{\pi / 3}$ that is readily confirmed by using the first of Eqs. (10). We conclude that

$$
\begin{gather*}
\int \boldsymbol{U} \cdot \boldsymbol{Y}_{l m}^{*} d \Omega=\int \frac{\boldsymbol{u} \cdot \mathbf{\Psi}_{l m}^{*} d \Omega}{2}=\sqrt{\frac{2 \pi}{3}}\left(U_{x}+i U_{y}\right) \delta_{l 1} \delta_{m,-1} \\
-\sqrt{\frac{2 \pi}{3}}\left(U_{x}-i U_{y}\right) \delta_{l 1} \delta_{m, 1}+2 \sqrt{\frac{\pi}{3}} U_{z} \delta_{l 1} \delta_{m 0} \tag{114}
\end{gather*}
$$

We find the coefficients $c_{l m}$. We have from Eq. (26)

$$
\begin{align*}
& c_{l m}=\left(\frac{\lambda^{2}}{2}+\frac{3 \lambda K_{3 / 2}(\lambda)}{2 K_{1 / 2}(\lambda)}\right) \sqrt{\frac{2 \pi}{3}} \delta_{l 1}\left(\left(U_{x}+i U_{y}\right) \delta_{m,-1}\right. \\
& \left.-\left(U_{x}-i U_{y}\right) \delta_{m, 1}+\sqrt{2} U_{z} \delta_{m 0}\right) . \tag{115}
\end{align*}
$$

We observe from Eq. (16) that

$$
\begin{equation*}
K_{1 / 2}(\lambda)=\sqrt{\frac{\pi}{2 \lambda}} e^{-\lambda}, \quad \lambda K_{3 / 2}(\lambda)=\sqrt{\frac{\pi}{2 \lambda}}(1+\lambda) e^{-\lambda} .( \tag{116}
\end{equation*}
$$

We obtain

$$
\begin{align*}
& c_{l m}=\left(1+\lambda+\frac{\lambda^{2}}{3}\right) \sqrt{\frac{3 \pi}{2}} \delta_{l 1}\left(\left(U_{x}+i U_{y}\right) \delta_{m,-1}\right. \\
& \left.-\left(U_{x}-i U_{y}\right) \delta_{m, 1}+\sqrt{2} U_{z} \delta_{m 0}\right) \tag{117}
\end{align*}
$$

We find from Eq. (6) that the pressure is

$$
\begin{align*}
& p=\frac{3}{4 r^{2}}\left(1+\lambda+\frac{\lambda^{2}}{3}\right)\left(\left(U_{x}+i U_{y}\right) \sin \theta e^{-i \phi}+\left(U_{x}-i U_{y}\right)\right. \\
& \left.\times \sin \theta \exp (i \phi)+2 U_{z} \cos \theta\right)=\left(1+\lambda+\frac{\lambda^{2}}{3}\right) \frac{3 \boldsymbol{U} \cdot \boldsymbol{r}}{2 r^{3}}, \tag{118}
\end{align*}
$$

where we used $P_{1}^{1}(\cos \theta)=-\sin \theta$. For finding the velocity field it remains to obtain $\tilde{c}_{l m}^{r}$ since $\tilde{c}_{l m}=0$ by $\int \boldsymbol{u} \cdot \boldsymbol{\Phi}_{l m}^{*} d \Omega=0$, see the last of Eqs. (20). We have from Eq. (28) that

$$
\begin{align*}
& \tilde{c}_{l m}^{r}=-\frac{3 \int \boldsymbol{u} \cdot \boldsymbol{Y}_{l m}^{*} d \Omega}{\lambda K_{1 / 2}(\lambda)}=\left(-\sqrt{3}\left(U_{x}+i U_{y}\right) \delta_{l 1} \delta_{m,-1}\right. \\
& \left.+\sqrt{3}\left(U_{x}-i U_{y}\right) \delta_{l 1} \delta_{m, 1}-\sqrt{6} U_{z} \delta_{l 1} \delta_{m 0}\right) \frac{2 \exp (\lambda)}{\sqrt{\lambda}} \tag{119}
\end{align*}
$$

We find using Eqs. (8) and (118) that the flow is given by

$$
\begin{equation*}
\boldsymbol{u}=\boldsymbol{u}_{s}-\nabla\left(1+\lambda+\frac{\lambda^{2}}{3}\right) \frac{3 \boldsymbol{U} \cdot \boldsymbol{r}}{2 \lambda^{2} r^{3}} \tag{120}
\end{equation*}
$$

where $\boldsymbol{u}_{s}$ is a solution of the vector Helmholtz equation which according to Eqs. (18) and (116) is given by

$$
\begin{align*}
& \boldsymbol{u}_{s}=\sum_{m=-1}^{1}\left(\frac{\tilde{c}_{1 m}^{r} K_{3 / 2}(\lambda r) \hat{r} Y_{1 m}}{r^{3 / 2}}\right.  \tag{121}\\
& \left.-\frac{\tilde{c}_{1 m}^{r} \lambda r^{1 / 2} K_{1 / 2}(\lambda r)}{2}\left(1+\frac{1}{\lambda r}+\frac{1}{(\lambda r)^{2}}\right) \nabla Y_{1 m}\right)
\end{align*}
$$

The dependence on the coefficients reduces to $\sum_{m=-1}^{1} \tilde{c}_{1 m}^{r} Y_{1 m}$ as seen by rewriting the above as

$$
\begin{align*}
& \boldsymbol{u}_{s}=\frac{\hat{\boldsymbol{r}}}{r^{2}} \sqrt{\frac{\pi}{2 \lambda}}\left(\frac{1}{\lambda r}+1\right) e^{-\lambda r} \sum_{m=-1}^{1} \tilde{c}_{1 m}^{r} Y_{1 m}  \tag{122}\\
& -\frac{\lambda^{1 / 2}}{2} \sqrt{\frac{\pi}{2}} e^{-\lambda r}\left(1+\frac{1}{\lambda r}+\frac{1}{(\lambda r)^{2}}\right) \nabla \sum_{m=-1}^{1} \tilde{c}_{1 m}^{r} Y_{1 m} .
\end{align*}
$$

We find from Eqs. (26)-(28) that

$$
\begin{align*}
& \frac{\tilde{c}_{1 m}^{r}}{c_{1 m}}=-\frac{6}{\lambda^{2}\left(3 K_{3 / 2}(\lambda)+\lambda K_{1 / 2}(\lambda)\right)} \\
& =-\sqrt{\frac{2}{\pi \lambda}} \frac{2 \exp (\lambda)}{1+\lambda+\lambda^{2} / 3} \tag{123}
\end{align*}
$$

We conclude that

$$
\begin{align*}
& \sum_{m=-1}^{1} \tilde{c}_{1 m}^{r} Y_{1 m}=-\sqrt{\frac{2}{\pi \lambda}} \frac{2 \exp (\lambda)}{1+\lambda+\lambda^{2} / 3} \sum_{m=-1}^{1} c_{1 m} Y_{1 m} \\
& =-\sqrt{\frac{2}{\pi \lambda}} \frac{2 \exp (\lambda) r^{2} p}{1+\lambda+\lambda^{2} / 3}=-\sqrt{\frac{2}{\pi \lambda}} 3 \exp (\lambda) \boldsymbol{U} \cdot \hat{\boldsymbol{r}}, \tag{124}
\end{align*}
$$

where we used Eqs. (6) and (118). We obtain using the above in Eq. (122) that

$$
\begin{align*}
& \boldsymbol{u}_{s}=-\left(\frac{1}{\lambda r}+1\right) \frac{3 \exp (\lambda(1-r))(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{2} \lambda}  \tag{125}\\
& +\frac{3 \exp (\lambda(1-r))}{2}\left(1+\frac{1}{\lambda r}+\frac{1}{(\lambda r)^{2}}\right) \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r}
\end{align*}
$$

which can be rewritten as

$$
\begin{align*}
& \boldsymbol{u}_{s}=\frac{3 \exp (-\lambda(r-1))}{2 \lambda^{2}}  \tag{126}\\
& \times\left(\frac{(1+\lambda r)(\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{r^{3}}+\frac{\lambda^{2}(\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{r}\right) .
\end{align*}
$$

We conclude that the flow induced by an oscillating sphere is given by $l=1$ term of the series solution, similarly to the counterpart problem for the steady Stokes flow.

## A Comparison with known solution

We compare the above with the solution brought in [1] which can be written as $(\hat{\boldsymbol{r}} \equiv \boldsymbol{r} / r)$

$$
\begin{align*}
& \boldsymbol{u}=\frac{3}{2 \lambda^{2}}\left(1+\lambda+\frac{\lambda^{2}}{3}+\frac{1}{\lambda^{2}}\left(e^{\lambda}-1-\lambda-\frac{\lambda^{2}}{3}\right) \nabla^{2}\right)(1 \\
& \times\left((1-(1+\lambda r) \exp (-\lambda r)) \frac{2(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right. \\
& \left.+\left(\left(1+\lambda r+(\lambda r)^{2}\right) \exp (-\lambda r)-1\right) \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right) .
\end{align*}
$$

We remark that [1] has a typo of multiplicative factor $\lambda$. We observe that Laplacian of terms that do not contain the exponential factors is proportional to

$$
\begin{equation*}
\nabla^{2} \frac{3 \hat{r}_{i} \hat{r}_{k}-\delta_{i k}}{r^{3}}=\nabla^{2} \nabla_{i} \nabla_{k} \frac{1}{r}=0 \tag{128}
\end{equation*}
$$

The Laplacian of the exponential terms is proportional to

$$
\begin{align*}
& e^{\lambda r} \nabla^{2} e^{-\lambda r}\left((1+\lambda r) \frac{\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}+(\lambda r)^{2} \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right) \\
& =\left(\lambda^{2}-\frac{2 \lambda}{r}\right)\left((1+\lambda r) \frac{\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}+(\lambda r)^{2} \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right) \\
& -2 \lambda \partial_{r}\left((1+\lambda r) \frac{\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}+(\lambda r)^{2} \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right) \\
& +\nabla^{2}\left((1+\lambda r) \frac{\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}+(\lambda r)^{2} \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right) . \tag{129}
\end{align*}
$$

We find taking the derivatives

$$
\begin{align*}
& e^{\lambda r} \nabla^{2} e^{-\lambda r}\left((1+\lambda r) \frac{\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}+(\lambda r)^{2} \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right) \\
& =\left(\lambda^{2}-\frac{2 \lambda}{r}\right)\left((1+\lambda r) \frac{\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}+(\lambda r)^{2} \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right) \\
& +2 \lambda\left((3+2 \lambda r) \frac{\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{4}}+(\lambda r)^{2} \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{4}}\right) \\
& -\frac{4 \lambda(\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{r^{4}}-2 \lambda^{2} \frac{\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}} \\
& =\frac{\lambda^{2}(1+\lambda r)(\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{r^{3}}+\frac{\lambda^{4}(\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{r} \tag{130}
\end{align*}
$$

We find collecting the terms that

$$
\begin{align*}
& \boldsymbol{u}=\frac{3}{2 \lambda^{2}}\left(1+\lambda+\frac{\lambda^{2}}{3}\right)\left((1-(1+\lambda r) \exp (-\lambda r)) \frac{2(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right. \\
& \left.+\left(\left(1+\lambda r+(\lambda r)^{2}\right) \exp (-\lambda r)-1\right) \frac{\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{r^{3}}\right) \\
& +\frac{3}{2 \lambda^{2}}\left(e^{\lambda}-1-\lambda-\frac{\lambda^{2}}{3}\right) e^{-\lambda r} \\
& \times\left(\frac{(1+\lambda r)(\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{r^{3}}+\frac{\lambda^{2}(\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{r}\right) \\
& =\frac{3}{2 \lambda^{2}}\left(1+\lambda+\frac{\lambda^{2}}{3}\right) \frac{3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}-\boldsymbol{U}}{r^{3}}+\frac{3 \exp (-\lambda(r-1))}{2 \lambda^{2}} \\
& \times\left(\frac{(1+\lambda r)(\boldsymbol{U}-3(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{r^{3}}+\frac{\lambda^{2}(\boldsymbol{U}-(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}})}{r}\right) . \tag{131}
\end{align*}
$$

It is seen that the final formula is rather simple and $\boldsymbol{u}(r=$ $1)=\boldsymbol{U}$ is readily checked. We can write the solution given by Eq. (131) in the form given by Eq. (8)

$$
\begin{equation*}
\boldsymbol{u}=\boldsymbol{u}_{s}-\frac{\nabla p}{\lambda^{2}}, \quad p=-\frac{3}{2}\left(1+\lambda+\frac{\lambda^{2}}{3}\right)(\boldsymbol{U} \cdot \nabla) \frac{1}{r} \tag{132}
\end{equation*}
$$

where $\boldsymbol{u}_{s}$ is a solution of the Helmholtz equation which can be readily seen to agree with Eq. (126). We conclude that the solution by the series and the ordinary solutions agree.

We remark that the solution can also be written in concise form which can be inferred from 30, 40]

$$
\begin{align*}
\boldsymbol{u} & =(\boldsymbol{U} \cdot \nabla) \nabla \psi-\boldsymbol{U} \nabla^{2} \psi \\
& \psi=\frac{3}{2 \lambda^{2} r}\left(1+\lambda+\frac{\lambda^{2}}{3}-\exp (\lambda(1-r))\right) \tag{133}
\end{align*}
$$

that can be readily confirmed to agree with other forms. This form expresses the solution via a single scalar function $\psi$ of the radial variable similarly to [3]. We have

$$
\begin{equation*}
\boldsymbol{u}_{s}=\left(\boldsymbol{U} \nabla^{2}-(\boldsymbol{U} \cdot \nabla) \nabla\right) \frac{3 \exp (\lambda(1-r))}{2 \lambda^{2} r} \tag{134}
\end{equation*}
$$

## Appendix D OSCILLATORY ROTATIONS OF THE SPHERE

In this Appendix the details of the calculations for the oscillatory rotations of the sphere, considered in subsection VC are provided. It is readily seen that

$$
\begin{aligned}
& \int \sin \theta \cos \phi Y_{l m}^{*} d \Omega=\pi\left(\delta_{m, 1}-\delta_{m,-1}\right) \sqrt{\frac{(2 l+1)}{4 \pi l(l+1)}} \\
& \times \int P_{l}^{1}(\cos \theta) \sin ^{2} \theta d \theta, \quad \int \cos \theta Y_{l m}^{*} d \Omega=2 \sqrt{\frac{\pi}{3}} \delta_{l 1} \delta_{m 0}
\end{aligned}
$$

where we have used

$$
\begin{equation*}
Y_{l 1}=\sqrt{\frac{(2 l+1)}{4 \pi l(l+1)}} P_{l}^{1}(\cos \theta) \exp (i \phi)=-Y_{l,-1}^{*} . \tag{135}
\end{equation*}
$$

We observe that the definition of $P_{l}^{m}$ implies

$$
\begin{align*}
& \int \sqrt{1-x^{2}} P_{l}^{1}(x) d x=-\int\left(1-x^{2}\right) P_{l}^{\prime}(x) d x \\
& =-2 \int x P_{l}(x) d x=-\frac{4 \delta_{l 1}}{3} \tag{136}
\end{align*}
$$

which gives

$$
\begin{align*}
\int \sin \theta \cos \phi Y_{l m}^{*} d \Omega & =\sqrt{\frac{2 \pi}{3}} \delta_{l 1}\left(\delta_{m,-1}-\delta_{m, 1}\right) \\
\int \sin \theta \sin \phi Y_{l m}^{*} d \Omega & =i \sqrt{\frac{2 \pi}{3}} \delta_{l 1}\left(\delta_{m,-1}+\delta_{m, 1}\right) \tag{137}
\end{align*}
$$

We conclude from the above and Eq. (20) that

$$
\begin{align*}
& -K_{3 / 2}(\lambda) \tilde{c}_{l m}=\int Y_{l m}^{*} \omega_{r} d \Omega=\sqrt{\frac{2 \pi}{3}}\left(\omega_{x}+i \omega_{y}\right) \delta_{l 1} \delta_{m,-1} \\
& -\sqrt{\frac{2 \pi}{3}}\left(\omega_{x}-i \omega_{y}\right) \delta_{l 1} \delta_{m, 1}+2 \sqrt{\frac{\pi}{3}} \omega_{z} \delta_{l 1} \delta_{m 0} \tag{138}
\end{align*}
$$

This gives on using Eq. (18) and the definition of $\boldsymbol{\Phi}$ in Eq. (9) that

$$
\begin{align*}
& \boldsymbol{u}=-\frac{K_{3 / 2}(\lambda r)}{K_{3 / 2}(\lambda)} \sqrt{\frac{2 \pi}{3 r}} \boldsymbol{r} \times \nabla\left(\left(\omega_{x}+i \omega_{y}\right) Y_{1,-1}\right.  \tag{139}\\
& \left.-\left(\omega_{x}-i \omega_{y}\right) Y_{11}+\sqrt{2} \omega_{z} Y_{10}\right)
\end{align*}
$$

We find by using the definitions of $Y_{1 m}$ and $Y_{1,-1}=-Y_{11}^{*}$ that the expression in brackets equals $\boldsymbol{\omega} \cdot \hat{\boldsymbol{r}}$. Finally using the definition of $K_{3 / 2}(\lambda)$ in Eq. (116) we obtain Eq. (68).

## Appendix E FORCE AND TORQUE VIA THE EXPANSION COEFFICIENTS

In this Appendix, we provide the formulae for the force and the torque on the sphere similar to those for the general solution of steady Stokes equations [1].

## A Difference from steady Stokes equations

For steady Stokes equations, that express the condition of zero total force on any volume of inertialess fluid, the force on the sphere of any radius is the same. This is because the total force on the volume enclosed by any two spheres vanishes. This leads to the possibility of finding the force from asymptotic behavior of the flow on an infinitely remote sphere. Thus the general series solutions of the Stokes equations provide the force as a single coefficient of the series that provides the leading order term at large distances. Similar fact holds for the torque 1].

In the case of the unsteady flow the situation is somewhat different and we have for the force $\boldsymbol{F}$ that the fluid exerts on interior of the unit sphere

$$
\begin{align*}
& F_{i}=\int_{r=1} \sigma_{i r} d S=\int_{x=R} \sigma_{i r} d S-\int_{1<r<R} \nabla_{k} \sigma_{i k} d V \\
& =\int_{r=R} \sigma_{i r} d S-\lambda^{2} \int_{1<r<R} u_{i} d V \tag{140}
\end{align*}
$$

where the stress tensor $\sigma_{i k}$, defined in Eq. (63), obeys $\nabla_{k} \sigma_{i k}=\lambda^{2} u_{i}$. The last, volume term, is proportional to the frequency and absent for steady Stokes flows.

## B Force and torque from reciprocal theorem

We can circumvent the direct calculation by using reciprocal theorem [1] similarly to 32]. We observe that for any dual flow obeying $\lambda^{2} \hat{v}_{i}=\nabla_{k} \hat{\sigma}_{i k}$ and incompressibility condition we have the Lorentz identity

$$
\begin{equation*}
\hat{v}_{i} \frac{\partial \sigma_{i k}}{\partial x_{k}}=u_{i} \frac{\partial \hat{\sigma}_{i k}}{\partial x_{k}}, \quad \frac{\partial\left(\hat{v}_{i} \sigma_{i k}\right)}{\partial x_{k}}=\frac{\partial\left(u_{i} \hat{\sigma}_{i k}\right)}{\partial x_{k}} \tag{141}
\end{equation*}
$$

Here $\hat{\sigma}_{i k}$ is the stress tensor of the dual flow defined similarly to Eq. (63). We use as the dual flow the oscillating
sphere solution considered in Sec. $\overline{\text { We find by integra- }}$ tion of the above identity over the sphere exterior that

$$
\boldsymbol{U} \int_{r=1} \sigma \hat{r} d S=-\int_{r=1} \boldsymbol{u} \cdot \frac{3(1+\lambda) \boldsymbol{U}+\lambda^{2}(\boldsymbol{U} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}}}{2} d S
$$

where we used Eq. (64). This equation must hold for any $\boldsymbol{U}$ providing us with a simple formula for the force

$$
\begin{equation*}
\boldsymbol{F}=-\frac{3(1+\lambda)}{2} \int_{r=1} \boldsymbol{u} d S-\frac{\lambda^{2}}{2} \int_{r=1}(\boldsymbol{u} \cdot \hat{\boldsymbol{r}}) \hat{\boldsymbol{r}} d S \tag{142}
\end{equation*}
$$

We conclude that the most general dependence of the force on $\lambda$ is parabolic. The parabola has only two free coefficients, $\boldsymbol{F}=\lambda^{2} \boldsymbol{F}_{1}+(1+\lambda) \boldsymbol{F}_{2}$ where $\boldsymbol{F}_{i}$ are frequency-independent and given by low moments of velocity field. This is the same dependence on $\lambda$ as for oscillating sphere, see Eq. (65). The only difference is that there are six free coefficients in the force and not three. The interpretation of the terms and their form in time domain are the same as for the oscillating sphere problem, see e.g. [1, 3].

We can similarly obtain the torque. We use as the dual flow the oscillatory rotation of the sphere considered in the previous section. We have

$$
\int_{r=1} \sigma \hat{r} \cdot(\boldsymbol{\omega} \times \hat{\boldsymbol{r}}) d S=-\int_{r=1} \boldsymbol{u} \cdot(\boldsymbol{\omega} \times \hat{\boldsymbol{r}})\left(3+\frac{\lambda^{2}}{1+\lambda}\right) d S
$$

where we used Eq. (69). We find the torque $\boldsymbol{T}$ by demanding that this equation holds for any $\boldsymbol{\omega}$

$$
\begin{equation*}
\boldsymbol{T}=-8 \pi \boldsymbol{\omega}_{e f f}-\frac{8 \pi \boldsymbol{\omega}_{e f f} \lambda^{2}}{3(1+\lambda)} \tag{143}
\end{equation*}
$$

where we introduced the effective angular velocity $\boldsymbol{\omega}_{\text {eff }}$ as

$$
\begin{equation*}
\boldsymbol{\omega}_{e f f} \equiv \frac{3}{8 \pi} \int_{r=1} \hat{\boldsymbol{r}} \times \boldsymbol{u} d S \tag{144}
\end{equation*}
$$

so that the torque formula looks as that for oscillating rotations of a rigid sphere, see Eq. (70). Thus the torque on the sphere generated by a general surface flow can be described as resulting from rigid sphere rotations.

## C Integrals' calculation

The above integral formulas for the force and the torque can be written in terms of the coefficients of the expansion in the VSH. This is done by inserting into the integrals the series expansion for $\boldsymbol{u}$. The calculation demands integrals of the VSH over the sphere. We consider

$$
\begin{align*}
& \int \boldsymbol{Y}_{l m} d \Omega=\hat{\boldsymbol{x}} \int \sin \theta \cos \phi Y_{l m} d \Omega+\hat{\boldsymbol{y}} \int \sin \theta \sin \phi Y_{l m} d \Omega \\
& +\hat{\boldsymbol{z}} \int \cos \theta Y_{l m} d \Omega \tag{145}
\end{align*}
$$

where we projected $\boldsymbol{Y}_{l m}=\hat{\boldsymbol{r}} Y_{l m}$ on the Cartesian unit vectors. We find using Eqs. (138) that

$$
\begin{align*}
& \int \boldsymbol{Y}_{l m} d \Omega=\delta_{l 1} \sqrt{\frac{2 \pi}{3}}\left(\hat{\boldsymbol{x}}\left(\delta_{m,-1}-\delta_{m, 1}\right)-i \hat{\boldsymbol{y}}\left(\delta_{m,-1}+\delta_{m, 1}\right)\right. \\
& \left.+\hat{\boldsymbol{z}} \sqrt{2} \delta_{m 0}\right) \tag{146}
\end{align*}
$$

This formula allows to find the first term in Eq. (140). It is readily seen from the general soluton that at large distances, considered in detail later,

$$
\begin{equation*}
p=\sum_{m=-1}^{m=1} \frac{c_{1 m} Y_{1 m}(\theta, \phi)}{r^{2}}+O\left(1 / r^{3}\right) \tag{147}
\end{equation*}
$$

and velocity is of order $1 / r^{3}$. Thus the integral of the surface traction over the surface of the sphere at infinity is fully determined by the pressure. The viscous component of the stress does not contribute in contrast with the steady Stokes equations. We have

$$
\begin{align*}
& \lim _{R \rightarrow \infty} \int_{x=R} \sigma_{i r} d S=-\sum_{m=-1}^{m=1} c_{1 m} \int \hat{\boldsymbol{r}} Y_{1 m}(\theta, \phi) d \Omega \\
& =-\sqrt{\frac{2 \pi}{3}} \sum_{m=-1}^{m=1}\left(\hat{\boldsymbol{x}}\left(c_{1,-1}-c_{11}\right)-i \hat{\boldsymbol{y}}\left(c_{1,-1}+c_{11}\right)\right. \\
& \left.+\hat{\boldsymbol{z}} \sqrt{2} c_{10}\right) \tag{148}
\end{align*}
$$

where we used Eq. (146) with $\hat{r} Y_{1 m}=\boldsymbol{Y}_{1 m}$. We consider the remaining integrals

$$
\begin{align*}
& \int \boldsymbol{\Psi}_{l m} d \Omega=\hat{\boldsymbol{x}} \int \hat{\boldsymbol{x}} \cdot \boldsymbol{\Psi}_{l m} d \Omega+\hat{\boldsymbol{y}} \int \hat{\boldsymbol{y}} \cdot \boldsymbol{\Psi}_{l m} d \Omega \\
& +\hat{\boldsymbol{z}} \int \hat{\boldsymbol{z}} \cdot \boldsymbol{\Psi}_{l m} d \Omega \tag{149}
\end{align*}
$$

and similar integrals for $\boldsymbol{\Phi}_{l m}$. Performing calculations similar to those that led to Eq. (146) we obtain

$$
\begin{align*}
& \int \boldsymbol{\Psi}_{l m} d \Omega=2 \delta_{l 1} \sqrt{\frac{2 \pi}{3}}\left(\hat{\boldsymbol{x}}\left(\delta_{m,-1}-\delta_{m, 1}\right)-i \hat{\boldsymbol{y}}\left(\delta_{m,-1}+\delta_{m, 1}\right)\right. \\
& \left.+\hat{\boldsymbol{z}} \sqrt{2} \delta_{m 0}\right)=2 \int \boldsymbol{Y}_{l m} d \Omega, \quad \int \boldsymbol{\Phi}_{l m} d \Omega=0 \tag{150}
\end{align*}
$$

For finding the torque, given by Eq. (143), we must calculate

$$
\begin{align*}
& \int_{r=1} \hat{\boldsymbol{r}} \times \boldsymbol{\Psi}_{l m} d S=\hat{\boldsymbol{x}} \int(\sin \theta \sin \phi \hat{\boldsymbol{z}}-\cos \theta \hat{\boldsymbol{y}}) \cdot \boldsymbol{\Psi}_{l m} d \Omega \\
& \quad+\hat{\boldsymbol{y}} \int(\cos \theta \hat{\boldsymbol{x}}-\sin \theta \cos \phi \hat{\boldsymbol{z}}) \cdot \boldsymbol{\Psi}_{l m} d \Omega \\
& +\hat{\boldsymbol{z}} \int(\sin \theta \cos \phi \hat{\boldsymbol{y}}-\sin \theta \sin \phi \hat{\boldsymbol{x}}) \cdot \boldsymbol{\Psi}_{l m} d \Omega \tag{151}
\end{align*}
$$

and similar integral for $\boldsymbol{\Phi}_{l m}$. We find calculating the integrals

$$
\begin{align*}
& \int_{r=1}^{\hat{\boldsymbol{r}}} \times \mathbf{\Phi}_{l m} d S=-2 \delta_{l 1} \sqrt{\frac{2 \pi}{3}}\left(\hat{x}\left(\delta_{m,-1}-\delta_{m, 1}\right)\right. \\
& \left.-i \hat{\boldsymbol{y}}\left(\delta_{m,-1}+\delta_{m, 1}\right)+\hat{\boldsymbol{z}} \sqrt{2} \delta_{m 0}\right)=-2 \int_{r=1} \boldsymbol{Y}_{l m} d S, \\
& \int_{r=1} \hat{\boldsymbol{r}} \times \boldsymbol{Y}_{l m} d S=\int_{r=1} \hat{\boldsymbol{r}} \times \mathbf{\Psi}_{l m} d S=0 . \tag{152}
\end{align*}
$$

Finally for finding the force and the torque via the expansion coefficients we introduce the series representation for $\boldsymbol{u}$ into Eqs. (142) and (143) and perform term-by-term integration using the formulas above.

## D Force via expansion coefficients

We obtain by setting $r=1$ in Eq. (19) and using Eq. (116) that inserting $\boldsymbol{u}$ into Eqs. (142) gives

$$
\begin{align*}
& \boldsymbol{F}=-\sqrt{\frac{2 \pi}{3}}\left(\hat{\boldsymbol{x}}\left(c_{1,-1}-c_{11}\right)-i \hat{\boldsymbol{y}}\left(c_{1,-1}+c_{11}\right)+\hat{\boldsymbol{z}} \sqrt{2} c_{10}\right) \\
& +\pi(1+\lambda) \sqrt{\frac{\lambda}{3}} e^{-\lambda}\left(\hat{\boldsymbol{x}}\left(\tilde{c}_{1,-1}^{r}-\tilde{c}_{11}^{r}\right)-i \hat{\boldsymbol{y}}\left(\tilde{c}_{1,-1}^{r}+\tilde{c}_{11}^{r}\right)\right. \\
& \left.+\hat{\boldsymbol{z}} \sqrt{2} \tilde{c}_{10}^{r}\right) . \tag{153}
\end{align*}
$$

It is seen from Eq. (146) that the first line of the above equation equals to the force due to the pressure component's of the stress, $-\int_{r=1} p \hat{\boldsymbol{r}} d \Omega$, cf. subsection VIIIC, Thus the last two lines describe the viscous component's contribution to the force.

The correspondence to the results holding in the limit of the steady Stokes flow, $\lambda \rightarrow 0$, is seen by observing that Eq. (23) gives

$$
\begin{equation*}
\tilde{c}_{l m}^{r} \simeq-\frac{l+1}{\lambda^{2} K_{l+1 / 2}(\lambda)} c_{l m}+o(\lambda) \tag{154}
\end{equation*}
$$

We find from Eq. (153) and (116) that

$$
\boldsymbol{F} \simeq-\sqrt{6 \pi}\left(\hat{\boldsymbol{x}}\left(c_{1,-1}-c_{11}\right)-i \hat{\boldsymbol{y}}\left(c_{1,-1}+c_{11}\right)+\hat{\boldsymbol{z}} \sqrt{2} c_{10}\right) .
$$

This equation agrees with the formula implied by Lamb's solution of the steady Stokes equations, see e.g. [1, 2], which is

$$
\begin{align*}
& \boldsymbol{F}=-4 \pi \nabla\left(r \sum_{m=-1}^{m=1} c_{1 m} Y_{l m}(\theta, \phi)\right)=-\sqrt{6 \pi} \\
& \times\left(\hat{\boldsymbol{x}}\left(c_{1,-1}-c_{11}\right)-i \hat{\boldsymbol{y}}\left(c_{1,-1}+c_{11}\right)+\hat{\boldsymbol{z}} \sqrt{2} c_{10}\right) \tag{155}
\end{align*}
$$

where we used the identity.

$$
\begin{align*}
& \sum_{m=-1}^{m=1} c_{1 m} Y_{l m}(\theta, \phi)=\sqrt{\frac{3}{4 \pi}} c_{10} \cos \theta-\sqrt{\frac{3}{8 \pi}} \sin \theta \\
& \times\left(\left(c_{11}-c_{1,-1}\right) \cos \phi+i\left(c_{11}+c_{1,-1}\right) \sin \phi\right) . \tag{156}
\end{align*}
$$

Returning to the general case, Eq. (153) does not make it obvious that the force dependence on $\lambda$ (see Eq (142)) is parabolic since it involves $\lambda$-dependent coefficients. We rewrite the force via the frequency-independent coefficients $b_{1 m}$ and $d_{1 m}$ defined in Eq. (81). We observe by using the definition of $K_{1 / 2}(\lambda)$ in Eq. (28) that

$$
\begin{equation*}
\tilde{c}_{1 m}^{r}=e^{\lambda} \sqrt{\frac{2}{\pi \lambda}}\left(d_{1 m}-3 b_{1 m}\right) \tag{157}
\end{equation*}
$$

Similarly using Eq. (80) and the definition of $\mathcal{P}_{1}(x)$ in Eq.(15) we have

$$
\begin{equation*}
c_{1 m}=\frac{\lambda^{2} b_{1 m}+(1+\lambda)\left(3 b_{1 m}-d_{1 m}\right)}{2} . \tag{158}
\end{equation*}
$$

We find Eq. (89) by using the above equations in Eq. (153).

## E Torque via expansion coefficients

The torque is found by inserting the expansion of $\boldsymbol{u}$ in the VSH into Eq. (143) and making term-by-term integration. This gives

$$
\begin{align*}
& \boldsymbol{T}=6\left(1+\lambda+\frac{\lambda^{2}}{3}\right) \frac{\pi}{\sqrt{3 \lambda^{3}}} e^{-\lambda}\left(\hat{\boldsymbol{x}}\left(\tilde{c}_{1,-1}-\tilde{c}_{11}\right)\right. \\
& \left.-i \hat{\boldsymbol{y}}\left(\tilde{c}_{1,-1}+\tilde{c}_{11}\right)+\hat{\boldsymbol{z}} \sqrt{2} \tilde{c}_{10}\right) . \tag{159}
\end{align*}
$$

It is readily seen that in the zero-frequency limit the above reproduces the result for the steady Stokes flow 1]. We remark that it is useful to employ in the demonstration that

$$
\int \boldsymbol{u} \cdot \boldsymbol{\Phi}_{1 m}^{*} d \Omega=2 \tilde{c}_{1 m} K_{3 / 2}(\lambda) \sim 2 \tilde{c}_{1 m} \sqrt{\frac{\pi}{2 \lambda^{3}}}
$$

The general $\lambda$-dependence of the torque is obtained by observing that the usage of $K_{3 / 2}(x)=(1+$ $1 / x) \exp (-x) \sqrt{\pi /(2 x)}$ in Eq. (27) gives

$$
\begin{equation*}
\tilde{c}_{1 m}=-\frac{\exp (\lambda)}{2(1+\lambda)} \sqrt{\frac{2 \lambda^{3}}{\pi}} \int_{r=1} Y_{1 m}^{*}(\nabla \times \boldsymbol{u})_{r} d \Omega \tag{160}
\end{equation*}
$$

The usage of this equation in Eq. (159) reproduces Eq. (143) with

$$
\begin{align*}
& \boldsymbol{\omega}_{e f f}=\frac{1}{4} \sqrt{\frac{3}{2 \pi}}\left(\hat{\boldsymbol{x}}\left(e_{-1}-e_{1}\right)-i \hat{\boldsymbol{y}}\left(e_{-1}+e_{1}\right)+\hat{\boldsymbol{z}} \sqrt{2} e_{0}\right), \\
& e_{m} \equiv \int_{r=1} Y_{1 m}^{*}(\nabla \times \boldsymbol{u})_{r} d \Omega . \tag{161}
\end{align*}
$$

which provides explicit form of $\boldsymbol{\omega}_{\text {eff }}$ in Eq. (144) via projections on $Y_{l m}$.

## Appendix F AXIALLY SYMMETRIC CASE

In this Appendix we provide the details for the consideration of the axially symmetric case in Sec. VII, We observe using Eq. (25) and integration over $\phi$ that in the axially symmetric case

$$
\begin{align*}
& \frac{1}{\sqrt{\pi(2 l+1)}} \int Y_{l 0}^{*} \nabla_{s} \cdot \boldsymbol{u} d \Omega=2 \int P_{l}(\cos \theta) u \sin \theta d \theta \\
& +\int P_{l}(\cos \theta) \partial_{\theta}(\sin \theta v) d \theta \tag{162}
\end{align*}
$$

Performing integration by parts in the last term and using $\partial_{\theta}\left[P_{l}(\cos \theta)\right]=P_{l}^{1}(\cos \theta)$ we find

$$
\begin{align*}
& \frac{1}{\sqrt{\pi(2 l+1)}} \int Y_{l 0}^{*} \nabla_{s} \cdot \boldsymbol{u} d \Omega=2 \int P_{l}(\cos \theta) u \sin \theta d \theta \\
& -\int P_{l}^{1}(\cos \theta) v \sin \theta d \theta \tag{163}
\end{align*}
$$

The use of this formula in Eq. (26) gives

$$
\begin{align*}
& c_{l 0}=\frac{\lambda \sqrt{\pi(2 l+1)} \mathcal{P}_{l}\left(\lambda^{-1}\right)}{(l+1) \mathcal{P}_{l-1}\left(\lambda^{-1}\right)} \int_{-1}^{1} P_{l}^{1}(x) v(x) d x  \tag{164}\\
& +\frac{\lambda\left(l \mathcal{P}_{l}\left(\lambda^{-1}\right)+\lambda \mathcal{P}_{l-1}\left(\lambda^{-1}\right)\right) \sqrt{\pi(2 l+1)}}{(l+1) \mathcal{P}_{l-1}\left(\lambda^{-1}\right)} \int_{-1}^{1} P_{l}(x) u(x) d x .
\end{align*}
$$

where $u(x), v(x)$ are defined as $u(\theta=\arccos x)$ and $v(\theta=$ $\arccos x)$, respectively. We obtain Eq. (78) for $D_{l}$ from the definition. For finding $F_{l}$ we use

$$
\begin{equation*}
\tilde{c}_{l 0}^{r}=\sqrt{\frac{2 \lambda}{\pi}} \frac{\exp (\lambda)}{\mathcal{P}_{l}\left(\lambda^{-1}\right)}\left(\int u Y_{l 0}^{*} d \Omega-\frac{(l+1) c_{l 0}}{\lambda^{2}}\right) \tag{165}
\end{equation*}
$$

see Eqs. (23) and (16). We find

$$
\begin{align*}
& \tilde{c}_{l 0}^{r}=-\frac{\exp (\lambda)}{\mathcal{P}_{l-1}\left(\lambda^{-1}\right)} \sqrt{\frac{2(2 l+1)}{\lambda}}\left(\int_{-1}^{1} P_{l}^{1}(x) v(x) d x\right. \\
& \left.+l \int_{-1}^{1} P_{l}(x) u(x) d x\right) \tag{166}
\end{align*}
$$

We find using the above relations in Eq. (28) that

$$
\tilde{c}_{l m}^{r}=\sqrt{\pi(2 l+1)} \frac{\int Y_{l m}^{*} \nabla_{s} \cdot \boldsymbol{u} d \Omega-l \int_{-1}^{1} P_{l}(x) u(x) d x}{\lambda \mathcal{P}_{l-1}\left(\lambda^{-1}\right)}
$$

thus

$$
\begin{align*}
& c_{l m}=\frac{\lambda\left(l K_{l+1 / 2}(\lambda)+\lambda K_{l-1 / 2}(\lambda)\right)}{(l+1) K_{l-1 / 2}(\lambda)} \int \boldsymbol{u} \cdot \boldsymbol{Y}_{l m}^{*} d \Omega \\
& +\frac{\lambda K_{l+1 / 2}(\lambda)}{(l+1) K_{l-1 / 2}(\lambda)} \int \boldsymbol{u} \cdot \mathbf{\Psi}_{l m}^{*} d \Omega=\frac{\lambda^{2}}{l+1} \int \boldsymbol{u} \cdot \boldsymbol{Y}_{l m}^{*} d \Omega \\
& -\frac{\lambda^{2} K_{l+1 / 2}(\lambda) \tilde{c}_{l m}^{r}}{l+1} \tag{168}
\end{align*}
$$

We compare the general, not assuming any symmetry, solution given by Eq. (104) with solution of Rao [19] for
axially symmetric flow. For solutions vanishing at infinity it is given via the streamfunction $\psi$ is

$$
\psi=\sum_{l=1}^{\infty}\left(\frac{D_{l}(\lambda)}{r^{l}}+F_{l}(\lambda) r^{1 / 2} K_{l+1 / 2}(\lambda r)\right) \int_{-1}^{\cos \theta} P_{l}(x) d x
$$

where the coefficients $D_{l}$ and $F_{l}$ are determined by the boundary conditions. The flow is given by,

$$
u_{r}=\frac{1}{r^{2} \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad u_{\theta}=-\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}
$$

which gives,
The radial component of Eq. (104) is,

$$
\begin{equation*}
u_{r}^{0}=\sum_{l=1}^{\infty} \sum_{m=-l}^{l}\left(\frac{\tilde{c}_{l m}^{r} K_{l+1 / 2}(\lambda r)}{r^{3 / 2}}+\frac{l+1}{\lambda^{2} r^{l+2}}\right) c_{l m} Y_{l m}, \tag{169}
\end{equation*}
$$

where we used Eq. (9). In the axially symmetric case only the terms with $m=0$ are non-vanishing giving
$u_{r}^{0}=$
$\sum_{l=1}^{\infty}\left(\frac{\tilde{c}_{l 0}^{r} K_{l+1 / 2}(\lambda r)}{r^{3 / 2}}+\frac{l+1}{\lambda^{2} r^{l+2}}\right) c_{l 0} \sqrt{\frac{2 l+1}{4 \pi}} P_{l}(\cos \theta)$.
This agrees with the solution given by Eq. (76) in the axially symmetric case if we identify

$$
\begin{align*}
& D_{l}(\lambda)=-\frac{(l+1) c_{l 0}}{\lambda^{2}} \sqrt{\frac{2 l+1}{4 \pi}} \\
& F_{l}(\lambda)=-\tilde{c}_{l 0}^{r} c_{l 0} \sqrt{\frac{2 l+1}{4 \pi}} \tag{171}
\end{align*}
$$

We check the remaining component of the velocity,

$$
\begin{align*}
& u_{\theta}=\sum_{l=1}^{\infty}\left(\frac{l D_{l}(\lambda)}{r^{l+2}}-\frac{F_{l}(\lambda)}{r} \frac{d\left(r^{1 / 2} K_{l+1 / 2}(\lambda r)\right)}{d r}\right) \\
& \frac{1}{\sin \theta} \int_{-1}^{\cos \theta} P_{l}(x) d x \tag{172}
\end{align*}
$$

We use the identity 36],
$\int_{-1}^{\cos \theta} P_{l}(x) \frac{d x}{\sin \theta}=\int_{-1}^{1} P_{l}(x) \frac{d x}{\sin \theta}-\int_{\cos \theta}^{1} P_{l}(x) \frac{d x}{\sin \theta}$
$=\frac{\partial_{\theta} P_{l}(\cos \theta)}{l(l+1)}$.
where we used $\int_{-1}^{1} P_{l}(x) d x=0$ for $l>0$. Thus we can write,

$$
\begin{align*}
& u_{\theta}=\sum_{l=1}^{\infty}\left(\frac{l D_{l}(\lambda)}{r^{l+2}}-\frac{F_{l}(\lambda)}{r} \frac{d\left(r^{1 / 2} K_{l+1 / 2}(\lambda r)\right)}{d r}\right) \\
& \frac{\partial_{\theta} P_{l}(\cos \theta)}{l(l+1)} \tag{174}
\end{align*}
$$

The polar component of Eq. (104) in the axially symmetric case is,
$u_{\theta}^{0}=\sum_{l=1}^{\infty}\left(\frac{\tilde{c}_{l 0}^{r}}{l(l+1) r} \frac{d\left(r^{1 / 2} K_{l+1 / 2}(\lambda r)\right)}{d r}-\frac{1}{\lambda^{2} r^{l+2}}\right) c_{l 0}$
$\sqrt{\frac{2 l+1}{4 \pi}} \partial_{\theta} P_{l}(\cos \theta)$,
where we used Eq. (9). This agrees with Rao's solution using Eq. (171) completing the proof.


[^0]:    * itzhak8@gmail.com
    $\dagger$ lisha@technion.ac.il
    $\ddagger$ bru@stowers.org
    § izi@me.technion.ac.il

