Chaotic Oscillations and Noise Transformations in a Simple Dissipative System with Delayed Feedback

V. V. Zverev * and B. Y. Rubinstein †

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Abstract

We analyze the statistical behavior of signals in nonlinear circuits with delayed feedback in the presence of external Markovian noise. For the special class of circuits with intense phase mixing we develop an approach for the computation of the probability distributions and multitime correlation functions based on the random phase approximation. Both Gaussian and Kubo-Andersen models of external noise statistics are analyzed and the existence of the stationary (asymptotic) random process in the long-time limit is shown. We demonstrate that a nonlinear system with chaotic behavior becomes a noise amplifier with specific statistical transformation properties.

KEY WORDS: Dynamical systems; Markov process; delayed feedback; probability distributions; Lyapunov exponent.

1 INTRODUCTION

The routes to chaos and the general laws of chaotic motion have been intensely investigated in recent years both theoretically and experimentally with remarkable achievements [1]-[5]. It seems that one of the most interesting and significant questions in this field is the problem of noise influence on chaotic motion (see refs. [6]-[11] and the references in [4] and [5]). First, this is because random fluctuations can erode the fine structure of the chaotic motion and chaotic attractors [9],[10]. From the statistical point of view we must deal with the more general problem of fluctuation transformations in nonlinear systems.

In the present paper we consider a simple theoretical model of a circuit with a nonlinear element and delayed feedback as shown in Fig. 1. Let $X \equiv |X|e^{i\phi}$ be a slowly varying complex amplitude of a signal in this circuit. We assume that:

(i) The transformation in the nonlinear element (NLE) consists of the phase change $\phi \rightarrow \phi + \theta(|X|)$ [we restrict our consideration to the simple case $\theta(|X|) = \lambda|X|^{2k} + \theta_0, k = 1, 2, 3, \ldots$] and the dissipation (energy loss) $|X| \rightarrow \kappa|X|, \kappa < 1$.

(ii) The signal trips from the NLE through the delay line (DL) and then comes to the summation device Σ, where it interferes with the external signal (ES).

(iii) The average value of the ES is fixed (and equal to unit after renormalization).

(iv) $\xi(t)$ is the noise component of the external signal (NCES), $\langle\xi(t)\rangle = 0$ (the brackets $\langle\ldots\rangle$ mean the statistical average).

*Department of Higher Mathematics, Ural Polytechnical Institute, 620002, Sverdlovsk K-2, USSR.
†Institute of Metal Physics, 620219, Sverdlovsk GSP-170, USSR

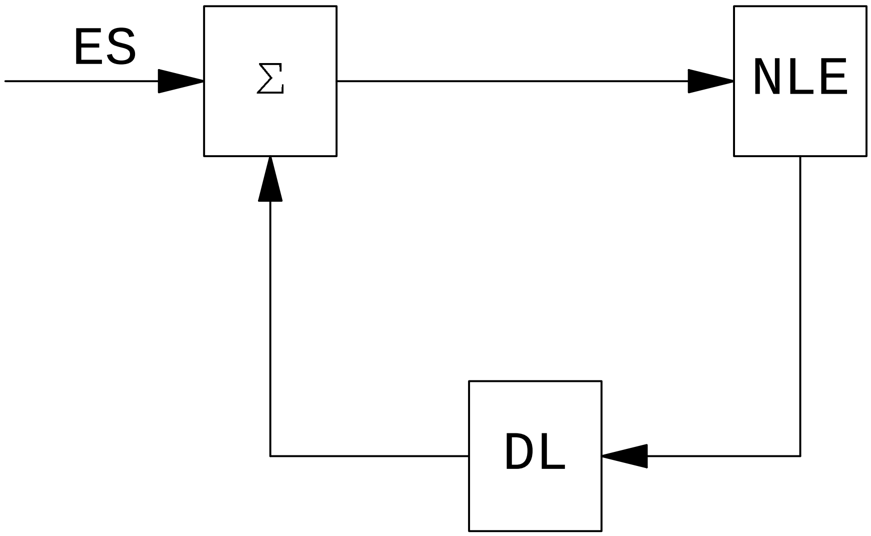

Figure 1: General scheme of the nonlinear circuit.

As a result, we have the equation of motion

$$X(t) = \xi(t) + 1 + \kappa X(t - T_d) \exp(i\lambda |X(t - T_d)|^{2k} + i\theta_0), \tag{1.1}$$

where T_d is the delay time (the round-trip time for the feedback loop); κ is the dissipation factor. Further, we will use the discrete time version of Eq. (1.1) (the evolutionary map):

$$X_{N+1} = \xi_N + F(X_N) \equiv \xi_N + 1 + \kappa X_N \exp(i\lambda |X_N|^{2k} + i\theta_0), \tag{1.2}$$

where $X_N = X(t_0 + NT_d)$ and $\xi_N = \xi[t_0 + (N+1)T_d]$, $0 \le t_0 < T_d$. For instance, Eqs. (1.1) and (1.2) describe some nonlinear electrical circuits and also the dynamic processes in the optical ring cavity containing the nonlinear medium (adiabatically) driven by the radiation [12] - [14] (Ikeda model). In this paper we focus on the case $\lambda \gg 1$ (intense phase mixing) and develop the statistical treatment for the dynamic behavior. We describe in Section 2 the case $T_d \gg \tau_c$, and in Section 3 the case $T_d \sim \tau_c$, where τ_c is the correlation time of the external noise. In Section 3 we also prove the existence of the asymptotic stochastic process in the long-time limit. Finally, in Section 4 we derive the simple approximate formulas for the maximal Lyapunov exponent.

2 STATISTICAL THEORY FOR $T_d \gg \tau_c$

In this section we develop the statistical theory for the case $T_d \gg \tau_c \gg \tau^*$, where τ^* is the relaxation (memory) time characterizing the rate of decay of excitations in the NLE. This assumption implies that the signals reaching the device Σ are statistically independent and $\langle \xi_N \xi_K \rangle = \langle \xi_N \xi_K^* \rangle = 0$ for $N \ne K$. Hence we can use the Kolmogorov-Chapman equation [15]

$$P_{N+1}(X) = \int\int dY dZ P_{\text{fl}}(X - Y) K(Y, Z) P_N(Z) \tag{2.1}$$

Here $P_N \equiv P(X[t_0 + NT_d])$ and $P_{\text{fl}}(X)$ are the probability distributions for the signal amplitude (at given time $t = t_0 + NT_d$) and for the external noise P_{fl}, respectively (we assume that the latter is a stationary random process and so P_{fl} is time independent). Also, $K(Y, Z) =$

$\delta^{(2)}(Y - F(Z))$, where $\delta^{(2)}(X) = \delta(\mathrm{Re}X)\delta(\mathrm{Im}X)$ is the two-dimensional δ-function and F was defined in (1.2). Setting

$$P_N(X) = \int dU \Theta_N(U) \exp[i \,\mathrm{Re}(XU^*)] \tag{2.2}$$

we find the equation for the Fourier transformations

$$\Theta_{N+1}(U) = \int dV \Theta_{\mathrm{fl}}(U) \rho(U,V) \Theta_N(V), \tag{2.3}$$

where

$$\rho(U,V) = (2\pi)^{-2} \int dY \exp[i \,\mathrm{Re}(YV^*) - i \,\mathrm{Re}(F(Y)U^*)] \tag{2.4}$$

and $\Theta_{\mathrm{fl}}(U)$ is the Fourier transformation of $P_{\mathrm{fl}}(X)$. In particular, for the Gaussian noise, $\Theta_{\mathrm{fl}}(U) = \exp(-R|U|^2/4)$, where $R = \langle \xi_N \xi_N^* \rangle$. Substituting F from Eq. (1.2) to Eq. (2.4) and making use of formulae for the Hankel transformation, we have

$$\rho(U,V) = \tilde{\rho}(U,V) + \Delta\rho(U,V) \tag{2.5}$$

where

$$\tilde{\rho}(ue^{i\alpha}, ve^{i\beta}) = (2\pi v)^{-1} \exp(-iu\cos\alpha)\delta(v - ku), \tag{2.6}$$

$$\Delta\rho(ue^{i\alpha}, ve^{i\beta}) = (4\pi)^{-1} \sum_{\nu \neq 0} W_\nu(u,v) \exp[i\nu(\theta_0 - \alpha + \beta) - iu\cos\alpha] \tag{2.7}$$

and

$$W_\nu(u,v) = \int_0^\infty J_\nu(v\sqrt{s}) J_\nu(\kappa u\sqrt{s}) e^{i\nu\lambda s^k} ds. \tag{2.8}$$

Using the standard stationary phase method [16], we can obtain the asymptotic evaluation for the integral in Eq. (2.8) for $\lambda \gg 1$ (u and v are constant). For this purpose (and in accordance with the localization principle [16]) we use the decomposition for unity into smooth functions: $1 = Q_1(s) + Q_2(s); Q_1(s) \equiv 0$ for $s \leq \delta_1, Q_1(s) \equiv 1$ for $s > \delta_2$, where $0 < \delta_1 < \delta_2$. Multiplying the Bessel functions in (2.8) by unity in this form, we obtain the sum of two integrals. It follows from the asymptotic properties of the Bessel functions that the integral containing Q_1 is $O(|\lambda|^{-\infty})$. Using the Erdèlyi's lemma for the other integral, we obtain the principal term of the asymptotic expansion:

$$W_\nu(u,v) \propto \frac{1}{k\nu!^2} (\nu\lambda)^{-\frac{\nu+1}{k}} \left(\frac{\kappa uv}{4}\right)^\nu \Gamma\left(\frac{\nu+1}{k}\right) e^{i\frac{\pi(\nu+1)}{2k}} \tag{2.9a}$$

Note that for $k = 1$

$$W_\nu(u,v) = \frac{1}{\nu\lambda} i^{\nu+1} J_\nu\left(\frac{\kappa uv}{2\lambda\nu}\right) \exp\left(-i\frac{v^2 + \kappa^2 u^2}{4\nu\lambda}\right), \tag{2.9b}$$

and the estimation (2.9a) follows from (2.9b), too.

Now let us discuss the connection between the NCES properties and the useful approximation for the kernel $\Theta_{\mathrm{fl}}\ \rho$ in Eq. (2.3). If $\lambda \gg 1$, then quantities W_ν are far from zero only for $u, v \gg 1$. This means that the contribution to ρ containing W_ν describes the fine structure of K (since K is the Fourier transform of ρ). Assuming that $\Theta_{\mathrm{fl}}(ue^{i\alpha})$ tends to zero rapidly enough as $u \to \infty$, one can see that the sum in Eq. (2.7) becomes negligible and $\Theta_{\mathrm{fl}}\ \rho \sim \Theta_{\mathrm{fl}}\ \tilde{\rho}$. It means qualitatively that the random fluctuations erode the fine structure of K (and therefore of P_N).

In addition to the above consideration, we can present the supplement for the case $k = 1$. Due to the asymptotic evaluation for the Bessel function,

$$|J_\nu(x)| \leq |J_\nu(j_{\nu,1})| = C_1 \nu^{-1/3} + 0(\nu^{-1})$$

(as $\nu \gg 1$) is valid (here $j_{\nu,1}$ denotes the first (left) maximum; see [20]), we have $\nu^{-1}|J_\nu(x)| \sim \nu^{-4/3}$ and $\sum_{\nu=1}^{\infty} |J_\nu(x)| < C_2$. As a result, using Eq. (2.9b), we have

$$\int |\Theta'_{N+1}(U)| dU \leq \frac{C_3}{\lambda} \left[\int |\Theta_{\text{fl}}(U)| dU \right] \left[\int |\Theta_N(V)| dV \right] \tag{2.10}$$

where

$$\Theta'_{N+1}(U) = \int dV \Theta_{\text{fl}}(U) \Delta\rho(U,V) \Theta_N(V)$$

and C_1, C_2, C_3 are constants. Assuming the convergence of the integrals in Eq. (2.10), we can see that the contribution $\Theta'_{N+1}(U)$ is negligible as $\lambda \gg 1$.

The replacement $\rho \to \tilde{\rho}$ in the Eq. (2.3) is equivalent to the replacement $K \to \tilde{K}$ in Eq. (2.1). Here

$$\tilde{K}(Y,Z) = \overline{\delta^{(2)}(Y - 1 - \kappa Z e^{i\eta})} = (2\pi|Y-1|)^{-1} \delta(|Y-1| - \kappa|Z|) \tag{2.11}$$

and η is a random variable with a uniform distribution in $[0, 2\pi]$ (here and henceforth an overbar means the phase average). Actually after these replacements we have a random irreversible evolutionary map instead of a deterministic one.

Supposing that the replacement $\rho \to \tilde{\rho}$ is valid, we shall distinguish two cases:
(i) the NCES intensity is small and so the statistical properties of a signal are independent of the NCES characteristics.
(ii) The noise component of the signal is the superposition of the NCES and the noise created by the chaotic motion.
First, we consider the case (i) and replace $\rho \to \tilde{\rho}$ in Eq. (2.3). The resulting approximate equation of motion reads

$$\Theta_{N+1}(U) = e^{-i \,\mathrm{Re}\, U} \; \overline{\Theta_N(\kappa|U|e^{i\phi})}, \tag{2.12}$$

and the equivalent equation for distributions takes the form

$$P_{N+1}(X) = \kappa^{-2} \; \overline{P_N(\kappa^{-1}|X-1|e^{i\phi})}. \tag{2.13}$$

Making the replacement $\Theta_{N+1}, \Theta_N \to \Theta_{st}$ in Eq. (2.12) we find the equation for the Fourier transformation of the stationary (invariant) distribution. We may write its solution in the form

$$\Theta_{st}(U) = (2\pi)^{-1} e^{-i \,\mathrm{Re}\, U} \prod_{\gamma=1}^{\infty} J_0(\kappa^\gamma |U|), \tag{2.14}$$

It easy to show the convergence of the infinite product in Eq. (2.14) and the stability of this solution. Likewise, $\Theta_{st} = \lim_{N\to\infty} \Theta_N$, where the series $\Theta_1, \ldots, \Theta_N$ is created starting from any absolutely integrable function Θ_1, using Eq. (2.13).

Note a peculiarity of the functional iteration process $P_N \to P_{N+1}$ defined by Eq. (2.13). To this aim let us suppose

$$P_N(X) = (\pi\kappa^2)^{-1} \; f_N(|X-1|^2/\kappa^2) \tag{2.15}$$

and get a new functional equation from Eq. (2.13):

$$f_{N+1}(G) = (\pi\kappa^2)^{-1} \int_{H_-}^{H_+} \frac{f_N(H) dH}{[-(H-H_-)(H-H_+)]^{1/2}}, \tag{2.16}$$

where $H_\pm = (1 \pm \sqrt{G})^2/\kappa^2$. Let $f_1(H) = 0$ for $H \notin [0, h_1]$. We have, after N iterations, $f_N(H) \equiv 0$ for $H \notin [h'_N, h_N]$, where $h_{N+1} = (1 + \kappa\sqrt{h_N})^2$ and $h'_{N+1} = (1 - \kappa\sqrt{h_N})^2$. We then obtain after simple calculations

$$\begin{aligned} h_\infty &= \lim_{N\to\infty} h_N &&= (1-\kappa)^{-2} \\ h'_\infty &= \lim_{N\to\infty} h'_N &&= \begin{cases} (1-2\kappa)^2/(1-\kappa)^2, & \text{for} \quad \kappa < 0.5 \\ 0, & \text{for} \quad 0.5 \le \kappa < 1 \end{cases} \end{aligned} \tag{2.17}$$

As a result, $f_{st}(H) = \lim_{N\to\infty} f_N(H) \equiv 0$ for $H \notin [h'_\infty, h_\infty]$, and we have for the corresponding two-dimensional distribution $P_{st}(X) \neq 0$ only within the ring domain for

$$(1-2\kappa)\kappa/(1-\kappa) \le |X-1| \le \kappa(1-\kappa) \quad \text{for} \quad \kappa < 0.5 \tag{2.18a}$$

or the disk domain

$$0 \le |X-1| \le \kappa(1-\kappa) \quad \text{for} \quad 0.5 \le \kappa < 1 \tag{2.18b}$$

In addition, the simple approximate formula for the stationary distribution holds true for $k \ll 1$:

$$P_{st}(X) = \begin{cases} \pi^{-2}[4\kappa^6 - (|X-1|^2 - \kappa^2)^2]^{-1/2} & \text{for} \quad ||X-1|^2 - \kappa^2| < 2\kappa^3 \\ 0 & \text{for} \quad ||X-1|^2 - \kappa^2| \ge 2\kappa^3 \end{cases} \tag{2.19}$$

The typical numerically computed phase portraits for the two-dimensional noiseless map [Eq. (1.2) with $\xi_N \equiv 0$] are shown in Fig. 2 (note the presence of uncontrollable "noise" resulting from the truncation errors in digital calculations). In Fig. 3 we show the radial section profile of the rotationally symmetric distribution $P_{st} = \lim_{N\to\infty} P_N$ found by the functional iterations $P_N \to P_{N+1}$ with the use of Eq. (2.13). The crosses in Fig. 3 represent data obtained by the numerical iterations $X_N \to X_{N+1}$. We can see that the randomlike distributions caused by the chaotic motion are in good agreement with the statistical treatment results (even if the noise is absent).

Now we turn to the case (ii). Replacing $\rho \to \tilde{\rho}$ in Eq. (2.12), we take $\Theta_{\rm fl}$ in the Gaussian form and obtain

$$\Theta_{st}(U) = (2\pi)^{-1} \exp\left(-i \,\mathrm{Re}\, U - \frac{R|U|^2}{4(1-\kappa^2)}\right) \prod_{\gamma=1}^{\infty} J_0(\kappa^\gamma |U|), \tag{2.20}$$

Let also $\kappa \ll 1$. To obtain the roughest approximation for P_{st} in this case one should retain only the first factor in the infinite product in Eq. (2.20). This gives

$$P_{st}(X) \simeq (\pi R)^{-1}(1-\kappa^2) \exp[-(\kappa^2 + |X-1|^2)(1-\kappa^2)/R] I_0(2\kappa(1-\kappa^2)|X-1|/R). \tag{2.21}$$

[here and in the next formula $I_s(.)$ are the modified Bessel functions]. The more precise approximate expression [having the same accuracy as (2.19)] reads

$$\begin{aligned} P_{st}(X) \;\; &\simeq \;\; (\pi R)^{-1}(1-\kappa^2) \exp[-(\kappa^2 + |X-1|^2)(1-\kappa^2)/R] \sum_{s=-\infty}^{\infty} (-1)^s I_s(2\kappa^3(1-\kappa^2)/R) \times \\ & \quad I_s\left(\kappa(1-\kappa^2)|X-1|(\sqrt{1+2\kappa} + \sqrt{1-2\kappa})/R\right) \times \\ & \quad I_s\left(\kappa(1-\kappa^2)|X-1|(\sqrt{1+2\kappa} - \sqrt{1-2\kappa})/R\right). \end{aligned} \tag{2.22}$$

If $\kappa \ll 1$, then values of P_{st} are far from zero only within the ring domain; the mean radius of this domain is of the order of κ. The radial section profiles of the distribution P_{st} are shown in Fig. 4.

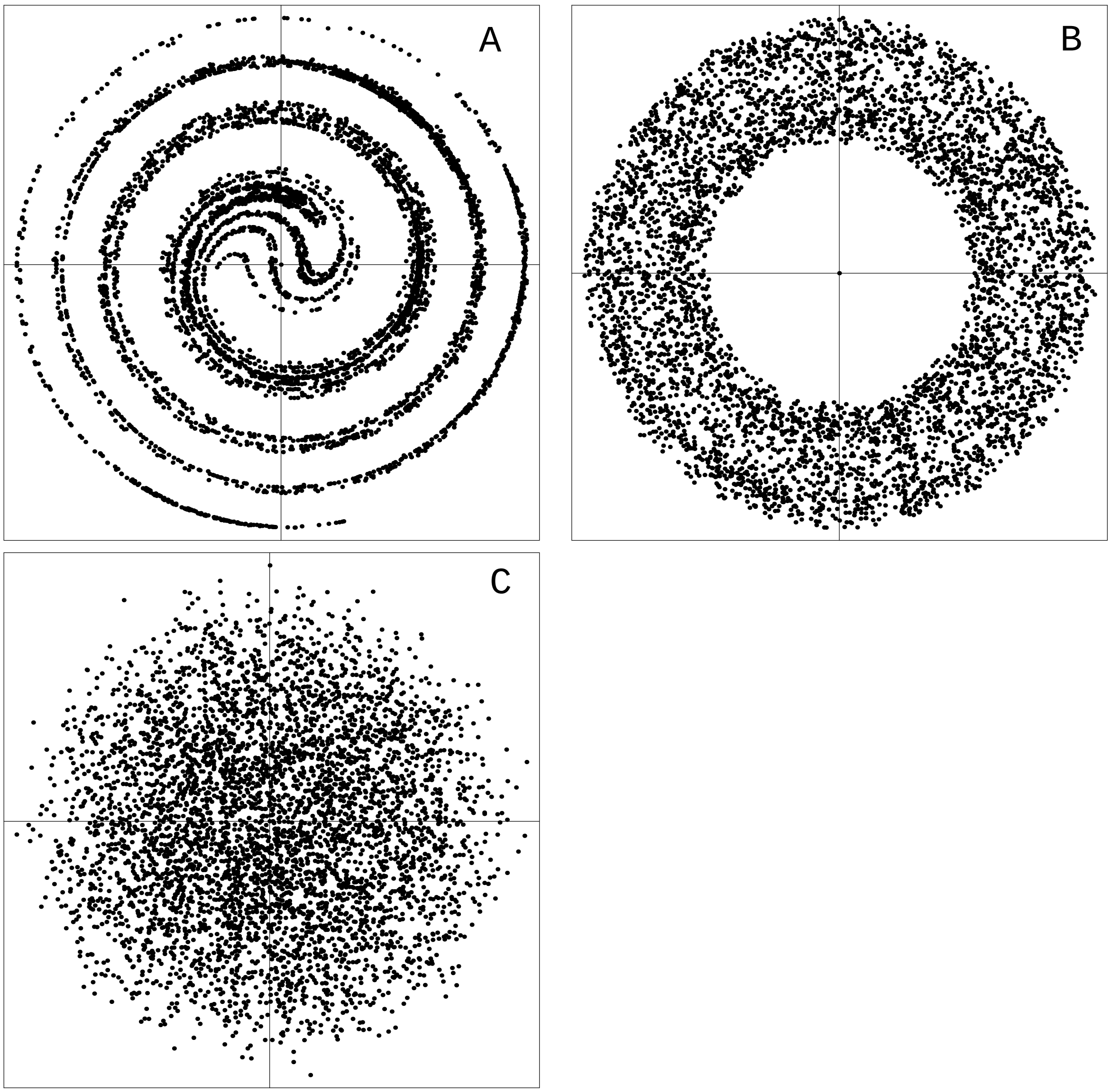

Figure 2: Typical chaotic attractors for the system described by Eq. (1.2), with $\xi_N \equiv 0$. The point distribution on the spiral Ikeda attractor (A) becomes randomlike (B,C) as $\lambda \gg 1$. The parameters are: (A) – $\kappa = 0.6, \lambda = 5$; (B) – $\kappa = 0.25, \lambda = 500$ (ring domain); (C) – $\kappa = 0.6, \lambda = 500$ (disk domain).

Fig. 2. (*Continued*)

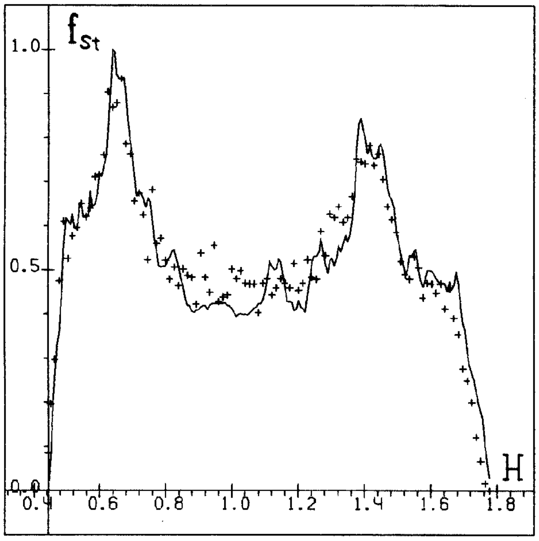

Figure 3: The radial section of the stationary probability distribution in the ring domain for $\kappa = 0.25$ (the noiseless case). The line shows the results of the numerical iterations $f_N \to f_{N+1}$ as $N \to \infty$ [calculated using Eq. (2.16)]. Crosses represent the analogous dependence obtained from the results of the numerical iterations $X_N \to X_{N+1}$ [see Eq. (1.2)] for $\lambda = 500, \kappa = 0.25$.

3 STATISTICAL THEORY FOR $T_d \sim \tau_c$. RANDOM PROCESS IN THE LONG-TIME ASYMPTOTICS

The situation $T_d \sim \tau_c \gg \tau^*$ is more difficult to analyze because one should take into account the existence of correlations between two fluctuating signals reaching the device Σ. In this section we suppose that the NCES is a Markovian stationary random process. This implies that multitime distributions (MTD) of the NCES are given by

$$P(\xi_n[t_n], \ldots, \xi_0[t_0]) = \omega(\xi_0) \prod_{p=0}^{n-1} \omega(\xi_{p+1}, \xi_p, [t_{p+1} - t_p]) \tag{3.1}$$

where $\omega(\xi) = \lim_{t\to\infty} \omega(\xi, \eta, [t])$, and $\xi_0, \ldots, \xi_n$ are the NCES amplitudes at given times $t_0, \ldots, t_n$ respectively. Now we can write the generalized Kolmogorov-Chapman equation for the MTD as

$$P_{N+1}((X_s), \xi_{n+1}) = \int d\mathbf{Y} d\boldsymbol{\xi} P_N((Y_s), \xi_0) \prod_{p=0}^{n} \delta^{(2)}(X_p - \xi_p - F(Y_p)) \omega(\xi_{p+1}, \xi_p, [\tau_{p+1} - \tau_p]), \tag{3.2}$$

In Eq. (3.2) we use the short notation defined by

$$P_N((X_s), \xi) = P((X_0[NT_d], X_1[NT_d + \tau_1], \ldots, X_n[NT_d + \tau_n]), \xi[NT_d]). \tag{3.3}$$

where $X_0, X_1, \ldots, X_n$ are the signal amplitudes at times $NT_d, NT_d + \tau_1, \ldots, NT_d + \tau_n$, respectively, and ξ is the NCES amplitude at a time NT_d (here and henceforth we assume that $0 \equiv \tau_0 < \tau_1 < \ldots < \tau_{n+1} \equiv T_d$; it follows that $NT_d \equiv NT_d + \tau_0$). We also use the notation $d\mathbf{A} = dA_0 dA_1 \ldots dA_n$. Using the Fourier transformation rule analogous to (2.2), we obtain

$$\Theta_{N+1}((U_s), \Omega_{n+1}) = \int d\mathbf{V} d\boldsymbol{\Omega} \Theta_N((V_s), \Omega_0) \prod_{p=0}^{n} \rho(U_p, V_p) H(\Omega_{p+1}, \Omega_p - U_p, [\tau_{p+1} - \tau_p]), \tag{3.4}$$

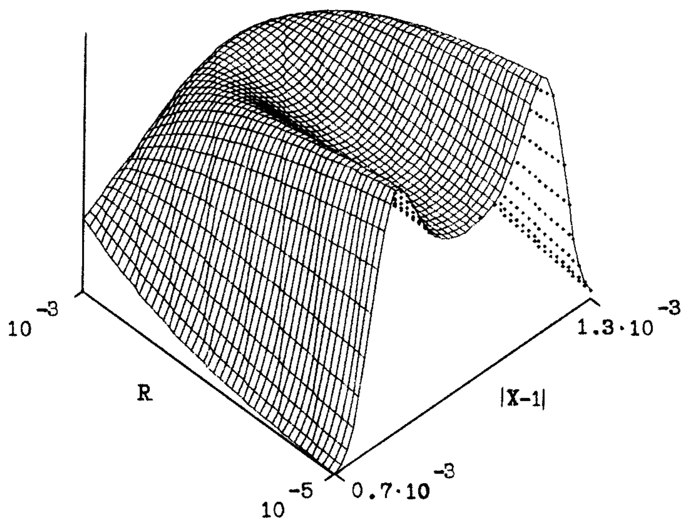

Figure 4: The radial section of the stationary probability distributions in the ring domain for various values of the Gaussian noise parameter R ($\kappa = 0.1$).

where

$$H(\Omega, \Gamma, [\tau]) = (2\pi)^{-2} \int d\xi \, d\eta \, \omega(\xi, \eta, [\tau]) \exp[i\mathrm{Re}(\eta\Gamma^* - \xi\Omega^*)] \tag{3.5}$$

and $\rho(U, V)$ was defined by Eq. (2.4). The short notation $\Theta_N((U_s), \Omega)$ can be unfolded in the same way as was done in Eq. (3.3) for the MTD.

In this paper we consider two models for the fluctuation statistics assuming the NCES to be either (a) a Gaussian random process (GP), or (b) a Kubo-Andersen random process (KAP). It should be mentioned that the latter, which is also called a generalized random telegraph process, is the stepwise constant Markovian process describing random jumps between (complex) values ξ_k (with appearance probability p_k); the jumping times are uniformly and independently distributed along the time axis [17]. The transition densities and their Fourier transformations are:
(a) for the GP case

$$\omega_{GP}(\xi, \eta, [\tau], R) = [\pi R(1 - \psi_\tau^2)]^{-1} \exp[-|\xi - \eta\psi_\tau|^2 / R(1 - \psi_\tau^2)], \tag{3.6}$$

$$H_{GP}(\Omega, \Gamma, [\tau], R) = \delta^{(2)}(\Gamma - \Omega\psi_\tau) \exp[-|\Omega|^2 R(1 - \psi_\tau^2)/4]; \tag{3.7}$$

(b) for the KAP case

$$\omega_{KAP}(\xi, \eta, [\tau]) = \psi_\tau \delta^{(2)}(\xi - \eta) + p(\xi)(1 - \psi_\tau), \tag{3.8}$$

$$H_{KAP}(\Omega, \Gamma, [\tau]) = \psi_\tau \delta^{(2)}(\Omega - \Gamma) + (1 - \psi_\tau)\chi(\Omega)\delta^{(2)}(\Gamma), \tag{3.9}$$

here

$$p(\xi) = \sum_{k=1}^{L} p_k \delta^{(2)}(\xi - \xi_k),$$

$$\chi(\Omega) = \sum_{k=1}^{L} p_k \exp[-i\mathrm{Re}(\xi_k \Omega^*)],$$

$$\psi_\tau = \exp(-\tau/\tau_c).$$

The replacement $\rho \to \tilde{\rho}$ [random phase approximation; see Eqs. (2.5)-(2.7)] leads to approximate equations of motion:
(a) for the GP case

$$\Theta_{N+1}((U_s), U_{n+1}) = \exp\left[-\frac{i}{2}\sum_{p=0}^{n}(U_p + U_p^*) - \Lambda((U_s), U_{n+1})\right] \times \overline{\Theta_N\left((\kappa U_s e^{i\phi_s}), \sum_{q=0}^{n+1} e^{-\tau_q/\tau_c} U_q\right)}, \tag{3.10}$$

(3.10) where

$$\Lambda((U_s), U_{n+1}) = (R/4)\sum_{p=1}^{n+1}\left|\sum_{q=p}^{n+1} e^{(\tau_p-\tau_q)/\tau_c} U_p\right|^2 (1 - e^{2(\tau_{p-1}-\tau_p)/\tau_c}) \tag{3.11}$$

and (b) for the KAP case

$$\Theta_{N+1}((U_s), U_{n+1}) = \exp\left[-\frac{i}{2}\sum_{p=0}^{n}(U_p + U_p^*) - \frac{T_d}{\tau_c}\right]\Bigg[\overline{\Theta_N((\kappa U_s e^{i\phi_s}), \beta_0^{n+2})}\times \sum_{\alpha=1}^{n+1}\sum_{\{k\}}\prod_{p=\{k\}}\left(e^{\frac{\tau_p-\tau_{p-1}}{\tau_c}} - 1\right) \times \prod_{\gamma=1}^{\alpha}\chi(\beta_{k_\gamma}^{k_\gamma+1})\,\overline{\Theta_N((\kappa U_s e^{i\phi_s}), \beta_0^{k_1})}\Bigg], \tag{3.12}$$

where $\beta_p^q = \sum_{i=p}^{q-1} U_i$; $\quad 1 \leq k_1 < \ldots < k_\alpha < k_{\alpha+1} = n+2$. The overbars in Eqs. (3.10) and (3.13) mean the averaging over all phase angles $\phi_s,\ s = 1, 2, \ldots, n$; $\{k\} \equiv k_1, k_2, \ldots, k_n$. We are also interested in multitime correlation functions (MTCF) of the form

$$m_N(p, q, (k_s, l_s)) = \left\langle \xi^p \xi^{*q} \prod_{i=0}^{n}(X_i^{k_i} X_i^{*l_i}) \right\rangle_N = \tag{3.13}$$

$$(2\pi)^{2n+4}(2i)^\gamma \left(\frac{\partial^p}{\partial\Omega^{*p}}\frac{\partial^q}{\partial\Omega^q}\right)\prod_{r=0}^{n}\left(\frac{\partial^{k_r}}{\partial U^{*k_r}}\frac{\partial^{l_r}}{\partial U^{l_r}}\right)\Theta_N((U_s), \Omega)\Big|_{\substack{\Omega=0\\ U_s=0}},$$

where $\gamma = p + q + \sum_{r=0}^{n}(k_r + l_r)$ is the order of the MTCF, and the brackets $\langle \ldots \rangle_N$ denote averaging with (3.3). Using Eqs. (3.10)-(3.13), we can obtain evolutionary maps for the MTCF. In order to shorten our account, we denote functions (3.14) as N-moments. It follows from the structure of the equations under consideration that every $(N+1)$-moment of the order γ is a linear function of N-moments of the order γ', where $\gamma' \leq \gamma$. Therefore, MTCF evolutionary maps are linear and finite-dimensional:

$$\mathbf{m}_{N+1} = \mathcal{A}\mathbf{m}_N + \mathbf{B} \tag{3.14}$$

where $\mathbf{m}_N$ is a vector composed of N-moments with $\gamma \leq \gamma_{\max}$, $\mathbf{B}$ is a fixed vector, and $\mathcal{A}$ is a square matrix.

Let us now investigate the long-time asymptotics of random signal motion in our circuit. Notice that the total description of the random process is provided with the infinite family of multitime correlation functions (moments). So it is sufficient to investigate the evolution of these functions only.

It follows from Eqs. (3.10)-(3.13) that any $(N+1)$-moment with fixed parameters (γ, α), $\alpha = p + q$, is a linear superposition of N-moments with parameters (γ', α') if either $\gamma' \leq \gamma$ or $\gamma' = \gamma, \alpha' \geq \alpha$. As a result, choosing the special form of a basis in the MTCF linear space, we can make the matrix $\mathcal{A}$ triangular. To achieve this, one must enumerate the basis elements (i.e., the moments with fixed parameters) in a special manner: if (γ_i, α_i) are the i-th element parameters, then $i > j$ means that either $\gamma_i > \gamma_j$ or $\gamma_i = \gamma_j,\ \alpha_i > \alpha_j$. The diagonal elements of the matrix $\mathcal{A}$ (which coincide with its eigenvalues) are:
(a) for the GP case

$$[\mathcal{A}](p, q, (k_s, l_s)) = \prod_{s=0}^{n} \delta_{k_s, l_s} \kappa^{k_s + l_s} e^{T_d(p+q)/\tau_c}, \tag{3.15}$$

(b) for the KAP case

$$[\mathcal{A}](p, q, (k_s, l_s)) = \prod_{s=0}^{n} \delta_{k_s, l_s} \kappa^{k_s + l_s} (\delta_{p0}\delta_{q0} + e^{T_d/\tau_c}(1 - \delta_{p0}\delta_{q0})), \tag{3.16}$$

Every diagonal element is less than unity. After the replacement $\mathbf{m}_{N+1}, \mathbf{m}_N \to \mathbf{m}_{st}$ we obtain the "stationary point" equation

$$\mathbf{m}_{st} = \mathcal{A}\mathbf{m}_{st} + \mathbf{B} \tag{3.17}$$

which has the only solution because $\mathcal{A}$ is a nondegenerate matrix. To prove the stability of this solution, one must consider the linearized equation for small deviations

$$\delta\mathbf{m}_{N+1} = \mathcal{A}\delta\mathbf{m}_N. \tag{3.18}$$

Let $\mathcal{A}$ be represented in the normal Jordan form $\mathcal{A} = TJT^{-1}$; here T is a nondegenerate matrix, and J is the Jordan matrix [18]. We have

$$\delta\mathbf{m}_{N+1} = \mathcal{A}^N \delta\mathbf{m}_0 = (TJT^{-1})^N \delta\mathbf{m}_0 = (TJ^N T^{-1})\delta\mathbf{m}_0 \tag{3.19}$$

We have from Eqs. (3.15) and (3.16) that the diagonal elements of a matrix J are also less than unity; therefore $J^N \to 0$ as $N \to \infty$ (see [18], p. 145). Hence, Eq. (3.19) leads to the conclusion $\delta\mathbf{m}_N \to 0$ as $N \to \infty$; this means the asymptotic stability of the "stationary point."

To complete our technique, we must show how to calculate the MTCF with arbitrary time values. It is easy to find that this function can be expressed in terms of the MTCF considered above. In view of the fact that general expressions are complicated and cumbersome, we write here only the two-time function (for the GP case):

$$\Theta(V[(N+k)T_d + \tau], U[NT_d], \Omega[NT_d]) = \overline{\exp[-i(VL_k + V^* L_k^*)/2]} \times$$
$$\overline{\exp\left(-\frac{R|V|^2}{4}\left\{[1 - \exp(-2T_d/\tau_c)](1 + \sum_{i=1}^{k-2}|G_i|^2) + [1 - \exp(-2\tau/\tau_c)]|G_{k-1}|^2\right\}\right)} \times$$
$$\overline{\Theta(\kappa^k V\{\exp(i\gamma)\}[NT_d + \tau], U[NT_d], \Omega + VG_{k-1}\{\exp(-\tau/\tau_c)\}[NT_d])}, \tag{3.20}$$

where

$$G_k = \exp(-kT_d/\tau_c)\left\{1 + \sum_{j=1}^{k}[\kappa \exp(T_d/\tau_c)]^j \exp(i\beta_j)\right\}$$

$$L_k = 1 + \sum_{j=1}^{k-1} \kappa^j \exp(i\beta_j)$$

and the overbar means averaging over phase angles $\beta_j, j = 1, \dots, k-1; \gamma$. Solving Eq. (3.17), one can obtain every MTCF in analytic form. Because the phase coherence is destroyed in the NLE, only the intensity correlation functions are of interest. Defining the covariance as

$$C(\tau) = \langle |X_{t+\tau}|^2 |X_t|^2 \rangle_{\text{st}} - \langle |X_{t+\tau}|^2 \rangle_{\text{st}} \langle |X_t|^2 \rangle_{\text{st}} \tag{3.21}$$

we have analytic expressions for various noise statistics:
(a) for the GP case

$$C(\tau) = \frac{1}{1-\kappa^4} \left(R^2 \frac{\psi_\tau^2 + \kappa^2 (\psi/\psi_\tau)^2}{1 - \kappa^2 \psi^2} + 2R \frac{\psi_\tau + \kappa^2 (\psi/\psi_\tau)}{1 - \kappa^2 \psi} \right), \tag{3.22}$$

$$C(kT_d + \tau) = \kappa^{2k} C(\tau) + 2R \frac{\psi \psi_\tau}{1 - \kappa^2 \psi} \sum_{l=0}^{k-1} \kappa^{2(k-l-1)} \psi^l + R^2 \frac{\psi^2 \psi_\tau^2}{1 - \kappa^2 \psi^2} \sum_{l=0}^{k-1} \kappa^{2(k-l-1)} \psi^{2l}, \tag{3.23}$$

(b) for the KAP case

$$C(\tau) = \frac{\psi_\tau + \kappa^2 (\psi/\psi_\tau)}{(1-\kappa^4)(1-\kappa^2\psi)} (\langle Q^2 \rangle_{KAP} - \langle Q \rangle_{KAP}^2), \tag{3.24}$$

$$C(kT_d + \tau) = \kappa^{2k} C(\tau) + \frac{\psi \psi_\tau}{1 - \kappa^2 \psi} \sum_{l=0}^{k-1} \kappa^{2(k-l-1)} \psi^l (\langle Q^2 \rangle_{KAP} - \langle Q \rangle_{KAP}^2). \tag{3.25}$$

Here $Q = |1+\xi|^2; 0 < \tau < T_d; \psi = \exp(-T_d/\tau_c); \psi_\tau = \exp(-\tau/\tau_c); \xi$ is a random variable; the brackets $\langle \dots \rangle_{KAP}$ mean averaging with the onedimensional distribution (for KAP). It is interesting to compare the aforementioned results with analogous ones for a linear dissipative circuit. Let a linear absorber takes the place of the NLE; accordingly we have to replace $F(X) \to F_L(X) = 1 + \kappa X e^{i\delta}$ in Eq. (1.2). The covariance for the GP statistics is

$$C(kT_r + \tau) = R|1-h|^{-2}(\Gamma_k + \Gamma_k^*) + R^2 |\Gamma_k|^2, \tag{3.26}$$

where

$$\Gamma_k = \frac{h^{*k}}{1-|h|^2} \left(\frac{\psi_\tau}{1-h\psi} - \frac{h^*(\psi/\psi_\tau)}{1-h^*\psi} \right) + \frac{\psi\psi_\tau}{1-h\psi} \sum_{l=0}^{k-1} (h^*)^{k-l-1} \psi^l \tag{3.27}$$

and $h = \kappa e^{-i\delta}$. Figure 5 shows the graphs of the normalized covariances (3.22)-(3.26).

4 APPROXIMATE FORMULAS FOR MAXIMAL LYAPUNOV EXPONENT

It is well known that the maximal Lyapunov exponent (MLE) characterizing the exponential divergence rate of initially close trajectories enables one to determine the type of system dynamics. If the motion is regular (a stable point or a limit cycle) the MLE is negative; a positive MLE corresponds to chaotic behavior.

In this section we present some simple approximate formulas for the MLE for the nonlinear system under investigation. Let $X_{N+1} = F(X_N)$ [see Eq. (1.2)]; the MLE can be obtained as [19]

$$L = \lim_{N\to\infty} \ln \sup |\sigma([\mathcal{M}_N \mathcal{M}_{N-1} \dots \mathcal{M}_1]^{1/N})|, \tag{4.1}$$

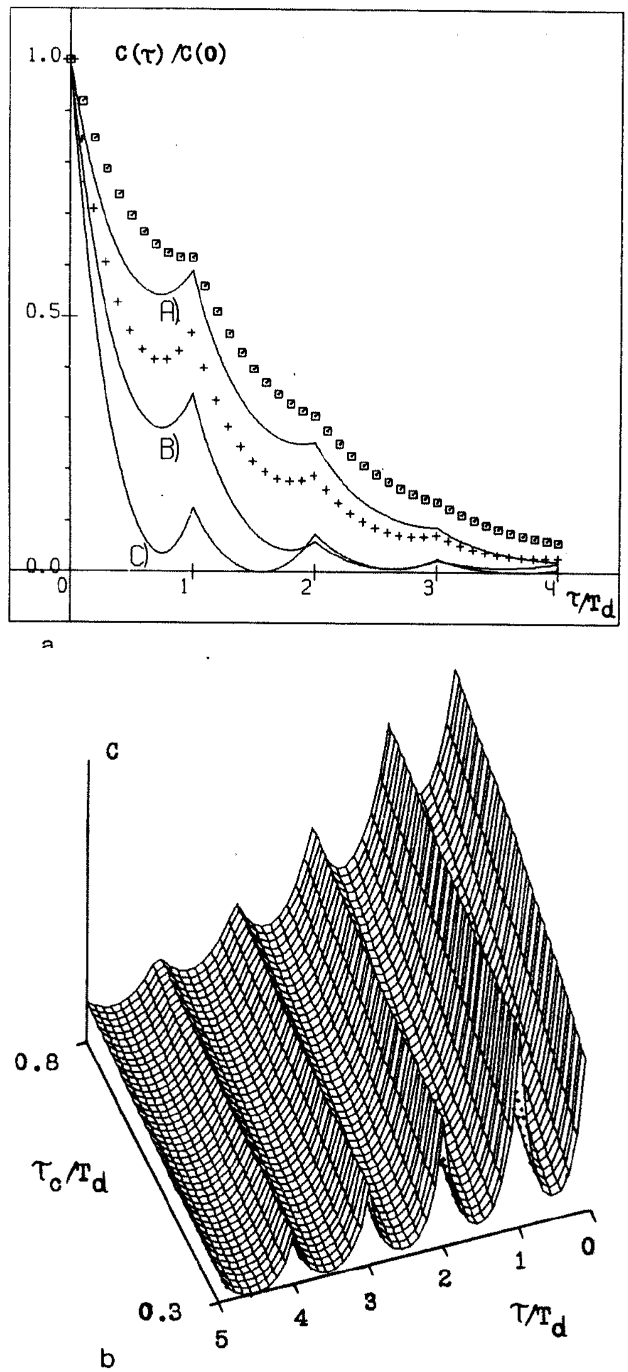

Figure 5: (a) Covariance of the signal intensities. Crosses and squares correspond to the circuit with NLE and the Gaussian noise statistics (crosses) or the Kubo-Andersen noise statistics (squares); lines are for the circuit with a linear absorber and the Gaussian noise, (A) $\delta = \pi/6$, (B) $\delta = \pi/2$,(C) $\delta = 5\pi/6$. (b) Two-time dependence graph for the intensity correlation function (τ and τ_c are normalized on fixed T_d).

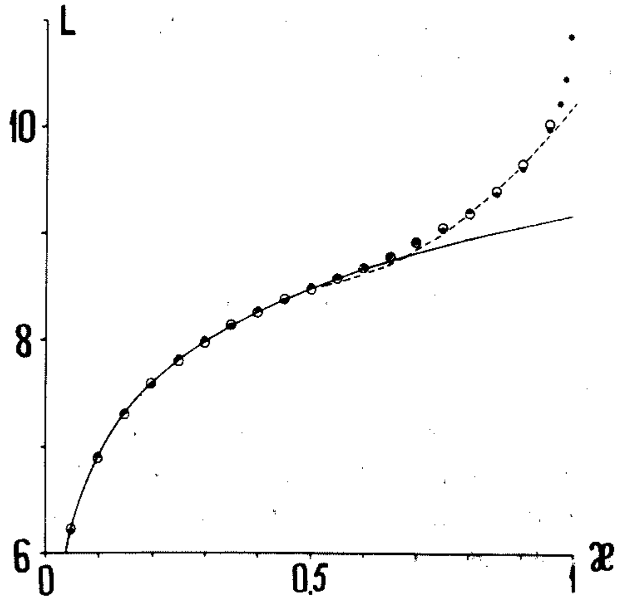

Figure 6: Maximal Lyapunov exponent vs. the dissipation parameter κ.

where $\mathcal{M}_p \equiv \mathcal{M}(X_p)$ is the Jacobian matrix calculated at point X_p; $\sigma(\mathcal{A})$ is the eigenvalue spectrum of $\mathcal{A}$. Using Eq. (1.2) ($\xi_N \equiv 0$), we determine

$$\mathcal{M}_p = \kappa \begin{pmatrix} \cos\gamma_p & -\sin\gamma_p \\ \sin\gamma_p & \cos\gamma_p \end{pmatrix} + 2\kappa\lambda k |Z_p|^{2k} \begin{pmatrix} -\sin(\phi_p+\gamma_p) \\ \cos(\phi_p+\gamma_p) \end{pmatrix} \times \begin{pmatrix} \cos\phi_p & \sin\phi_p \end{pmatrix}, \qquad (4.2)$$

where $\gamma_p = \lambda |X_p|^{2k} + \theta_0$, and $\phi_p = \arg X_p$ [the second term in Eq. (4.2) is the direct product of two-dimensional vectors]. It is natural to suppose that for $\lambda \gg 1$ the principal contribution is given by terms including λ in the maximal power. Keeping only this contribution and substituting expression (4.2) into Eq. (4.1), we obtain

$$L = \ln(2\kappa\lambda k) + \lim_{N\to\infty} N^{-1} \left(k \sum_{p=1}^{N} \ln|X_p|^2 + \sum_{p=1}^{N-1} \ln|\sin(\phi_{p+1} - \phi_p - \gamma_p)| + C \right), \qquad (4.3)$$

If we also take into account the relation

$$\sin(\phi_{p+1} - \phi_p - \gamma_p) = |Z_p|^{-1} \sin\arg(Z_p - 1)$$

and replace in Eq. (4.3) summation along the trajectory by integration with the invariant distribution, we have

$$L = \ln(\kappa\lambda k) + (2k-1)\theta(\kappa - 1/2) \int_{\kappa^{-2}}^{(1-\kappa)^{-2}} dH\, f_{st}(H) \ln(\kappa\sqrt{H}), \qquad (4.4)$$

where $\theta(x) = 0$ for $x \le 0$ and $\theta(x) = 1$ for $x > 0$; $f_{st} = \lim_{N\to\infty} f_N$, see Eq. (2.16). Notice that $f_{st}(H)$ (and the integral in the second term) is independent of k and λ; the second term vanishes for $\kappa < 0.5$. The plots of L vs. κ for $k = 1$ are shown in Fig. 6. Corresponding results were obtained numerically with the use of formula (4.4) (circles) and from datra calculated by iterations of the map (1.2) ($\xi_N \equiv 0$) (points). The dashed line corresponds to the simple approximation

$$L = \ln(\kappa\lambda) + \Theta(\kappa - 1/2)[(\kappa - 1/2)6.60 - 1.18](\kappa - 1/2), \tag{4.5}$$

and the solid line is the graph of $\ln(\kappa\lambda)$.

5 CONCLUSION

We have presented a statistical theory describing in the same manner the stochasticity caused by either deterministic chaos in a nonlinear system or ordinary fluctuations resulting from the influence of external noise. In this paper we have considered a nonlinear circuit with delayed feedback, and also the special case of intense phase mixing. The processes under investigation may be interpreted as noise amplification (generation) in nonlinear systems. Our description is based on the Kolmogorov-Chapman equations for the multitime distribution functions of the signal amplitude. We have developed a simple method of determining the multitime correlation functions of any order. Approximate formulas describing the dependence of the maximal Lyapunov exponent on the control parameters have been obtained, and are in good agreement with the results of numerical calculations.

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