Random self-modulation of radiation in a ring cavity. Case of strong mixing

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Abstract

It is shown that when certain conditions are met, the random dynamics of radiation in a ring cavity containing a nonlinear element can be described by means of a 2-D representation with a rapidly oscillating exponential. Such a representation generates strong mixing in the attracting region of the phase space and is equivalent to a certain random representation when the exciting signal contains a small noise component. Methods of finding the transient and stationary distribution densities in the attracting region are discussed, as is the maximum Lyapunov index characterizing the mixing rate.

References 1-3 have shown the fundamental possibility of formation of periodic and random self-oscillations of optical radiation in a ring cavity (RC) containing a nonlinear medium; subsequently, such oscillations have been observed experimentally [4]-[6]. Of interest are a further study and classification of the types of random oscillations (chaos), transformations of chaos, as well as scenarios of the genesis of chaos in RC and other nonlinear optical systems. This article is devoted to a study of developed chaos with strong mixing, which in a certain sense is the simplest type of chaos in RC and permits an analytical description based on the random-phase approximation.

Let us assume that a RC is excited by a partially coherent light wave through a beam splitter; after passage through the nonlinear element, part of the light leaves the RC through another beam splitter; no excitation of the counterpropagating wave takes place. Let us also assume that the nonlinear element is a cell with a two-level medium (TM) in which s-quantum transitions are excited ($s > 1$); the corresponding dynamic equations for a slowly changing amplitude E of the wave electric field, polarization P of the medium, and density n of population differences are:

$$\begin{aligned} E' + c^{-1}\dot{E} &= i\beta_s g_s P^*(E^*)^{s-1}, && (1)\\ \dot{P} + \frac{P}{T_2} - i(\omega_0 - s\omega)P &= ig_s(E^*)^s n, && (2)\\ \dot{n} + \frac{1+n}{T_1} &= i(g_s^* E^s P - c.c.)/2, && (3) \end{aligned}$$

where T_1 is the RC population relaxation time; T_2 is the RC phase mismatch time; ω_0 is the RC resonance frequency; ω is the frequency of the exciting field; the constants g_s and β_s are expressed in terms of the ordinary or composite (when $s > 1$) matrix elements of the transitions; the prime and dot above the letter, respectively, denote derivatives with respect to the coordinate z and time t. Let $\Delta = T_2(\omega_0 - s\omega) \gg 1$, so that the RC loss is low in comparison to the loss due to emission through the mirrors. We will also assume that $T_c \gg \tau_k \gg T_1, T_2$, where T_c is the round-trip time, and τ_k is the correlation time of the noise component of the exciting signal (NCE), which is a steady process. By virtue of the assumptions made and allowing for the boundary condition for the RC, we obtain [7, 8]

$$E(t) = \sqrt{1-\kappa}\, E_e(1+\xi(t)) + \kappa E(t-T_c)\exp[-i\chi_s(t) + i\theta], \tag{4}$$

where $\chi_s(t)$ is the phase change due to the nonlinear interaction of the light with the nonlinear medium; E_e and $E(t)$ are, respectively, the amplitudes of the coherent component of the exciting wave and of the wave at the entrance to the nonlinear medium (Im $E_e = 0$); $\xi(t)$ is the relative amplitude of the NCE; κ is the RC loss factor ($\kappa < 1$). Let us note that when $\xi = 0$, then using the inequality $|E(t)| \le \sqrt{1-\kappa}\, E_e + \kappa|E(t-T_c)|$, which follows from Eq. (4), one can show that the sequence $|E(t_0 + NT_c)|, N = 1, 2, \ldots$, is dominanted by a converging geometric progression, whence $|E(t)| \le E_e\sqrt{1-\kappa}$. This article is confined to the discussion of a special case that makes it possible to represent the dependence of the phase change on the field amplitude in the explicit and simplest form

$$\chi(t) = K_s B_s \Delta |E(t-T_c)|^{2s-2}(1 + B_s T_1 |E(t-T_c)|^{2s})^{-1} \tag{5}$$

Here and below $K_s + \beta_s l_0, B_s = |g_s|^2 T_2(1+\Delta^2)^{-1}$, and l_0 is the length of the nonlinear element. Equation (5) holds for all $s \ge 1$ if the adiabaticity condition $T_c \gg T_1, T_2$ applies, and when $s > 1$, in addition, the condition $K_s B_s E_e^{2s-2}(1-\kappa)^{-s} \ll 1$ holds (for more detail, see [7]). In order to simplify the analysis, we also make the usual assumption that the deviation of the population differences from the equilibrium value is small, this being valid when $B_s T_1 E_e^{2s}(1-\kappa)^{-s} \ll 1$, and we keep in Eq. (5) only the field-amplitude dependent term of the lowest order in the corresponding small parameter. Considering all of the above, we can describe the emission dynamics in the RC by means of the 2-D representation

$$Z_{N+1} = \xi_{N+1} + F(Z_N) \equiv \xi_{N+1} + 1 + \kappa Z_N \exp(i\lambda|Z_N|^{2k} + i\Omega) \tag{6}$$

where

$$\begin{aligned} &Z_N = E(t_0 + NT_c)/\sqrt{1-\kappa}E_e, \quad \xi_N = \xi(t_0 + NT_c,), \\ &\lambda = -K_1 B_1^2 T_1 \Delta(1-\kappa)E_e^2, \quad \Omega = \theta + K_1 B_1 \Delta, \quad k = 1 \ \text{ for } \ s = 1; \\ &\lambda = K_s B_s^2 \Delta(1-\kappa)^{s-1} E_e^{2s-2}, \quad \Omega = \theta, \quad k = s-1 \ \text{ for } \ s > 1. \end{aligned}$$

We set $\langle \xi_r \rangle = 0, \langle \xi_r \xi_l^* \rangle = R\delta_{rl}$, which is consistent with the condition imposed previously on the correlation time (angle brackets denote statistical averaging); here $R = I_n/E_c^2$, and I_n is the intensity of the NCE. Chaos with strong mixing, which will be of interest to us below, arises when $|\lambda| \gg 1$.

Figure 1 shows fragments of typical phase-plane diagrams of the emission dynamics in an RC, obtained by numerical iteration of the representation (6) when $k = 1, \xi_N = 0$ (let us note the presence of uncontrollable random perturbations caused by round-off errors). The upper (lower) sector is the image of 1/8 part of the distribution of the phase points in the attracting region of the Z plane (for convenience, in Fig. 1, the origin was shifted to the center of this region, located at the point $Z = 1$); the attracting region is in the shape of a ring (circle) when $\kappa = 0.2, \lambda = 400$ (when $\kappa = 0.8, \lambda = 100$). The results obtained clearly indicate the onset of dynamic stochastization in the RC. Assuming that $\tau_k \ll T_c$, and hence, the exciting wave and the wave which has completed a round trip of the RC are statistically independent, we describe the evolution of the random amplitude of the radiation field by means of the Kolmogorov-Chapman equation [9]:

$$P_{N+1}(X) = \int\int dY dZ P_n(X-Y)W(Y,Z)P_N(Z), \tag{7}$$

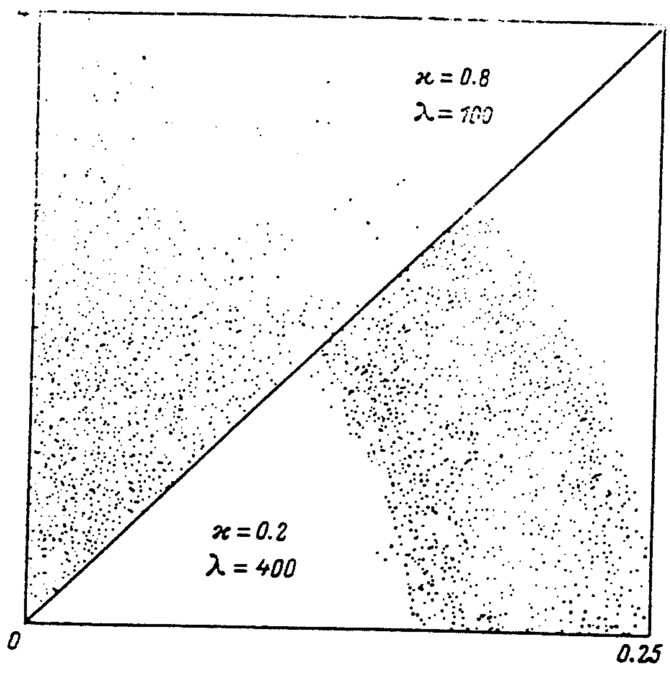

Figure 1: Fragments of distributions of phase points in the Z plane, obtained by numerical iteration of the representation (6).

where $P_N(X)$ is the density of the field amplitude distribution at time $t_0 + NT_c$; $P_n(X)$ is the distribution density of each of the independent random variables $\xi_r, r = 1, 2, 3, \ldots; W(Y, Z) = \delta^{(2)}(Y - F(Z))$, where $\delta^{(2)}(X) \equiv \delta(\mathrm{Re}X)\delta(\mathrm{Im}X)$ is the 2-D δ-function. Assuming

$$P_N(X) = \int dU \Psi_N(U) \exp[i\mathrm{Re}(XU^*)], \tag{8}$$

we turn to the equation for Fourier transforms

$$\Psi_{N+1}(U) = \int dV \Lambda(U) M(U, V) \Psi_N(V), \tag{9}$$

where

$$M(U, V) = (2\pi)^{-2} \int dY \exp[i\mathrm{Re}(YV^*) - i\mathrm{Re}(F(Y)U^*)], \tag{10}$$

and $\Lambda(U)$ is the characteristic function of the NCE; in the case of Gaussian noise $\Lambda_g(U) = \exp(-R|U|^2/4)$. Using Eq. (6), we find

$$M(ue^{i\alpha}, ve^{i\beta}) = \frac{1}{4\pi} e^{-iu\cos\alpha} \int_0^\infty db J_0 \left(\sqrt{b(v^2 + \kappa^2 u^2 - 2\kappa uv \cos(\lambda b^k + \Omega - \alpha + \beta)} \right). \tag{11}$$

With the aid of Graf's addition theorem and the representation of the δ-function resulting from the inversion theorem for the Hankel transformation [10], we represent Eq. (11) in the form

$$M(ue^{i\alpha}, ve^{i\beta}) = \tilde{M}(ue^{i\alpha}, ve^{i\beta}) + (4\pi)^{-1} \sum_{\nu \neq 0} \Phi_\nu(u, v) \exp[i\nu(\Omega - \alpha + \beta) - iu\cos\alpha], \tag{12}$$

where

$$\tilde{M}(ue^{i\alpha}, ve^{i\beta}) = \frac{1}{2\pi v} \exp(-iu\cos\alpha)\delta(v - \kappa u); \tag{13}$$

the functions Φ_ν are defined in the Appendix. Restricting the discussion to a nonrigorous treatment, let us consider the relationship between the intensity of the NCE and the form of the kernel of Eq. (12). According to the asymptotic estimates obtained in the Appendix, when the condition $\kappa uv|\lambda|^{-1/k} \ll 1$ applies, the integrals Φ_ν are small. It follows that if $|\lambda| \gg 1$, the terms containing Φ_ν, play an essential role only at large u and v, contributing to the small-scale structure of the kernel W and distributions P_N. In a case of nonzero NCE intensity with $|\Lambda(ue^{i\alpha})| \ll 1$ at large u, the r.h.s. of Eq. (9) includes the product of the function Λ and the kernel of Eq. (12), so that one can easily see that for a specific combination of Λ and $|\lambda|$, the contribution of the terms containing Φ_ν, may appear to be negligibly small. This corresponds to roughening of the density functions, *i.e.*, to loss of information on the small-scale structure (more rigorous analysis requiring a refinement of the asymptotic behavior of $\Lambda(ue^{i\alpha})$ as $u \to \infty$ is omitted in this treatment, which is confined to the assumption of a fairly rapid decrease). For Gaussian noise, the radius of a circle with the center $u = 0$ in which the function (9) differs appreciably from zero can be estimated at $2/\sqrt{R}$ of the halfwidth of the Gaussian Λ_g at the e^{-1} level. Accordingly, the condition of applicability of the approximation $M \to \tilde{M}$ will assume the form $R \gg \kappa|\lambda|^{-1/k}$, or

$$I_n K_1 B_1^2 T_1 \Delta(1-\kappa) \gg 1 \quad \text{for} \quad s = 1;$$
$$I_n (K_s B_s \Delta)^{1/(1-s)} \kappa^{-1} (1-\kappa) \gg 1 \quad \text{for} \quad s > 1.$$

The following cases will be distinguished: (a) the intensity of the NCE is low; it is sufficient for roughening of the distributions, but the specific statistical properties of the NCE do not affect the statistical characteristics of the emission in the RC; the latter are determined solely by the characteristics of the random dynamics perturbed by the noise; (b) the noise component of the emission in the RC results from the superposition of the NCE and of the noise generated by the chaos. In case (a) considered in this paper, the rough description is accomplished by means of the substitution $\Lambda M \to \tilde{M}$ in Eq. (9). Being applied to Eq. (7), the equivalent procedure consists in making the substitutions $P_n(X-Y) \to \delta^{(2)}(X-Y)$ and $W \to \tilde{W}$, where

$$\tilde{W}(Y,Z) = \langle \delta^{(2)}(Y - 1 - \kappa Z e^{i\eta}) \rangle = (2\pi|Y-1|)^{-1} \delta(|Y-1| - \kappa|Z|) \tag{14}$$

and leads to the equation

$$P_{N+1}(X) = \frac{1}{2\pi\kappa^2} \int d\phi \; P_N\left(\frac{|X-1|}{\kappa} e^{i\phi}\right). \tag{15}$$

Let us note that the substitution $W \to \tilde{W}$ in Eq. (7) is equivalent to the substitution $\lambda|Z_N|^{2k} \to \eta$ in Eq. (6), where η is a random phase uniformly distributed from 0 to 2π [the angle brackets in Eq. (4) denote averaging over this distribution]. The indicated substitution converts a deterministic representation to a random one and leads to loss of time reversibility. Since the r.h.s. of Eq. (15) depends on the only variable $|X-1|$, it is useful to make the substitution

$$P_N(X) = \frac{1}{\pi\kappa^2} f_N\left(\frac{|X-1|^2}{\kappa^2}\right), \tag{16}$$

Then Eq. (15) becomes

$$f_{N+1}(X) = \frac{1}{\pi\kappa^2} \int_{H_-}^{H_+} \frac{f_N(H)dH}{\sqrt{-(H-H_-)(H-H_+)}}, \tag{17}$$

where $H_\pm = (1 \pm \sqrt{G})^2/\kappa^2$. Using Fig. 2, we will elucidate the characteristics of the iteration process determined by the integral transformation (17). For a fixed G, the integration interval

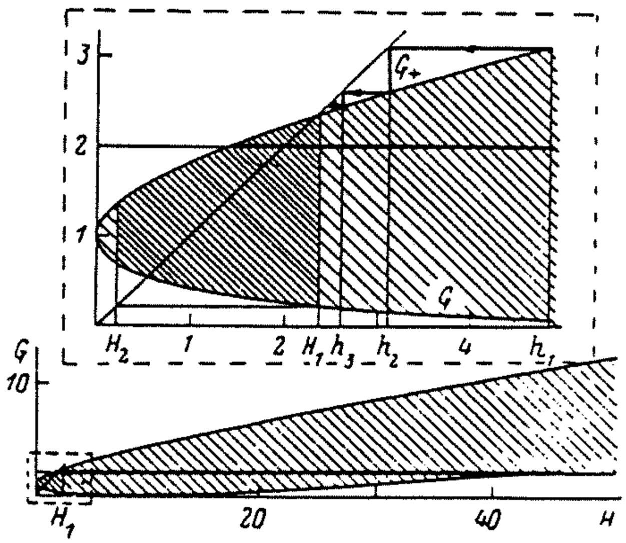

Figure 2: Integration interval in Eq. (17) is a segment of the the horizontal line located in shaded region (in the search for transition distributions) or in the dark-shaded region (in the search for a stationary distribution). Top shows enlarged image of rectangle bounded by shading lines.

will coincide with the portion of the horizontal line inside the shaded region, whose boundaries are $G_{\pm}(H) = (1 \pm \kappa\sqrt{H})^2$ (Fig. 2 corresponds to $\kappa = 0.35$). It can readily be seen that if $f_N(H) \equiv 0$ for $H > h_N$, then $f_{N+1}(H) \equiv 0$ for $H > h_{N+1}$, where $h_{N+1} = G_+(h_N)$. The iteration sequence $h_1, h_2, \ldots, h_N, \ldots$ converges to the solution of the equation $H = G_+(H)$, equal to $H_1 = (1-\kappa)^{-2}$, since the following conditions are satisfied: the function G_+ increases; $G_+(H) < H$ for $H > H_1$; $G_+(H) > H$ for $H < H_1$.Furthermore, if $f_N(H) \equiv 0$ outside the segment $[0, H_1]$, then $f_{N+1}(H) \equiv 0$ outside the segment $[H_2, H_1]$, where $H_2 = G_-(H_1) = (1-2\kappa)^2/(1-\kappa)^2$. Thus the functional iteration sequence $f_1, f_2, \ldots, f_N, \ldots$ starting from a finite function f_1, consists of finite functions; in the limit $N \to \infty$, the segment $[H_2, H_1]$ is the carrier. All of the above arguments remain in force for $\kappa < 0.5$; in the case $0.5 \leq \kappa < 1$, the graph of the function G is tangent to the abscissa axis to the left of point H_1, so that the limiting carrier is a segment $[0, H_1]$. The corresponding distributions in Z plane are described by Eq. (16) and have as carriers the ring at $\kappa < 0.5$ and the circle at $0.5 \leq \kappa < 1$ (see Fig. 1). It is important that not only the sequence of the carriers, but also that of the distribution densities converges to a limit. To prove this, it suffices to consider the Fourier transform of the stationary (invariant) distribution

$$\Psi_s(ue^{i\alpha}) = \frac{1}{2\pi} \exp\left(-u\cos\alpha\right) \prod_{\gamma=1}^{\infty} J_0(u\kappa^{\gamma}), \tag{18}$$

which is a solution of the equation obtained from Eq. (9) after the substitutions $\Lambda M \to \tilde{M}$; $\Psi_{N+1}, \Psi_N \to \Psi_s$; the convergence of the infinite product in the r.h.s. of Eq. (18) follows from the convergence of the series $|\Psi_s|$; the stationary solution is stable.

Figure 3 shows the results, obtained in different ways, of a numerical calculation of the limiting stationary distribution, $f_s(H)$ for $\kappa = 0.2$. The points represent the dependence calculated by means of Eq. (17); the initial distribution was chosen uniform on the segment

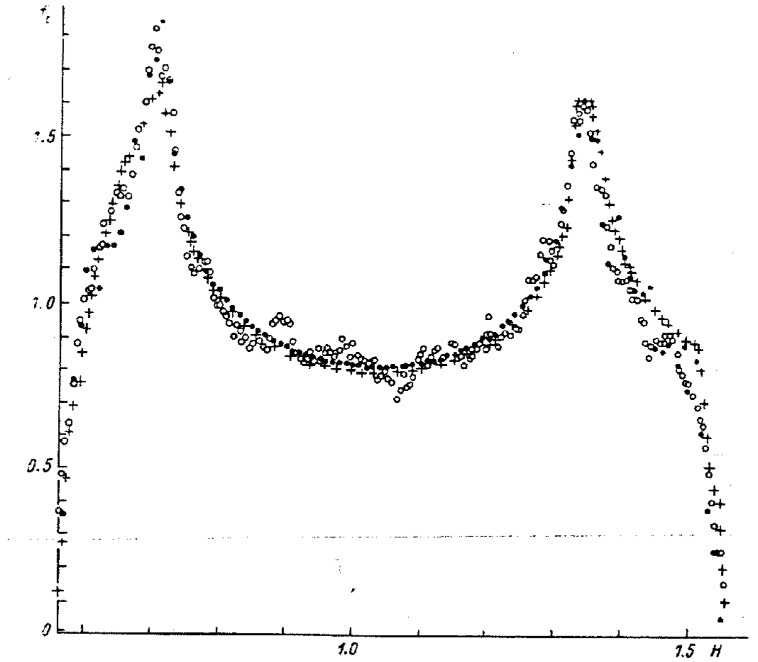

Figure 3: 1-D stationary distribution for $\kappa = 0.2$, calculated in different ways.

$[H_2, H_1]$ and equal to zero outside this segment; the integration intervals were segments of horizontal lines located in the dark shaded region in Fig. 2. The dependence shown by circles was found by iteration of the representation (6) (without noise, $\lambda = 10^4, k = 1$) and subsequent numerical smoothing of the small-scale structure. Crosses represent the results of the solution of the integral equation for f_s obtained from (17) after the substitution $f_{N+1}, f_N \to f_s$, by expansion in a Fourier series; the first 50 terms of the series were kept.

From a physical standpoint, a transformation of the statistical properties of the radiation and noise generation take place in the RC. For example, the distribution shown in Fig. 3 is associated with the superposition of a coherent wave of intensity $0.8E_e^2$ and a noise wave; the latter has a phase uniformly distributed between 0 and 2π, and an intensity distributed over the interval $(0.018 - 0.05)E_e^2$; the distribution has maxima at $0.022E_e^2$ and $0.042E_e^2$ (the intensity of the exciting wave is assumed to be equal to E_e^2). When $\lambda = 10^4, k = 1$, the approximate statistical description resulting in the distribution in question applies if $I_n \sim 10^{-4}E_e^2$.

We will distinguish, in particular, the case $\kappa \ll 1$. Retaining in the r.h.s. of Eq. (17) under the root sign only quantities containing κ to the lowest order, we can find the approximate expression

$$f_s(H) = \begin{cases} \frac{1}{\pi}[-(H-1-2\kappa)(H-1+2\kappa)]^{-1/2}, & \text{if} \quad |H-1| < 2\kappa; \\ 0, & \text{if} \quad |H-1| \geq 2\kappa \end{cases} \tag{19}$$

As the universal quantitative measure of randomness of the motion, it is usually used the rate of divergence of the initially close trajectories averaged along a trajectory, *i.e.*, the maximum Lyapunov index (MLI); a detailed discussion of the behavior of the MLI and of its relationship to the dimensional characteristics of an attracting set (attractor) is given in [11]. For a system described by the representation $Z_{N+1} = F(Z_N)$, the MLI can be calculated from the equation

$$L = \lim_{N\to\infty} \ln\sup|\sigma([M_N M_{N-1}\dots M_1]^{1/N})|, \tag{20}$$

where M_p is the Jacobi representation matrix, calculated at the point Z_p; the symbol $\sigma(A)$ denotes the spectrum of the matrix A. The MLI can usually be found only from the results of numerical modeling. We will show that for the MLI characterizing the iteration dynamics of the representation (6) for $|\lambda| \gg 1$, there exists a simple approximate formula. The Jacobi representation matrix for (6) without noise is

$$M_p = \kappa \begin{pmatrix} \cos\gamma_p & -\sin\gamma_p \\ \sin\gamma_p & \cos\gamma_p \end{pmatrix} + 2\kappa\lambda k|Z_p|^{2k} \begin{pmatrix} -\sin(\phi_p+\gamma_p) \\ \cos(\phi_p+\gamma_p) \end{pmatrix} \times \begin{pmatrix} \cos\phi_p & \sin\phi_p \end{pmatrix}, \tag{21}$$

where $\gamma_p = \lambda|Z_p|^{2k} + \Omega$ and $\phi_p = \arg Z_p$, [the second term in Eq. (21) is given by the direct product of 2-D vectors]. Substituting the matrices (21) into the matrix product entering into Eq. (20), and keeping only the term containing λ to the highest power, we obtain

$$L = \ln(2\kappa\lambda k) + \lim_{N\to\infty}\frac{1}{N}[k\sum_{p=1}^{N}\ln|Z_p|^2 + \sum_{p=1}^{N-1}\ln|\sin(\phi_{p+1}-\phi_p-\gamma_p)| + C], \tag{22}$$

where C denotes quantities independent of N and vanishing after taking the limit. Noticing that

$$\sin(\phi_{p+1}-\phi_p-\gamma_p) = |Z_p|^{-1}\sin\arg(Z_p-1)$$

and replacing the averaging over discrete time by averaging with the density of stationary distribution, we finally find

$$L = \ln(\kappa\lambda k) + (2k-1)\Theta(\kappa-1/2)\int_{\kappa^{-2}}^{(1-\kappa)^{-2}} dH f_s(H)\ln(\kappa\sqrt{H}), \tag{23}$$

where $\Theta(x) = 0$ when $x \le 0$; $\Theta(x) = 1$ when $x > 0$. We should point out that f_s (and hence, the integral in the second term) is independent of k and λ; the second term disappears if $\kappa < 0.5$. When $k = 1$ the dependence of L on λ and κ is satisfactorily described by

$$L = \ln(\kappa\lambda) + \Theta(\kappa-1/2)[(\kappa-1/2)6.60 - 1.18](\kappa-1/2), \tag{24}$$

In Fig. 4, the dependence of L on κ, shown by dots, was drawn using the results of numerical iteration of the representation (6) for $k = 1, \lambda = 10^4$. The circles represent the dependence obtained from Eq. (23) by using f_s found numerically. In this figure, the continuous and dashed lines represent plots of the functions $\ln\kappa\lambda$ and Eq. (24), respectively. Let us note in the conclusion that we discussed the 1-D variant of the theory based on an approximate 1-D evolutionary representation, obtained from Eq. (6) by the substitution $Z_N \to |Z_N|$ (for details see [7]).

APPENDIX

Using the standard stationary phase method [12], one can obtain asymptotic estimates for the integrals entering into Eq. (12)

$$\begin{aligned}\Phi_\nu(u,v) &= \int_0^\infty J_\nu(v\sqrt{S})J_\nu(\kappa u\sqrt{S})e^{i\nu\lambda S^k}dS \propto \\ &\frac{1}{k\nu!^2}(\nu\lambda)^{-\frac{\nu+1}{k}}\left(\frac{\kappa uv}{4}\right)^\nu\Gamma\left(\frac{\nu+1}{k}\right)e^{i\frac{\pi(\nu+1)}{2k}}\end{aligned} \tag{A1}$$

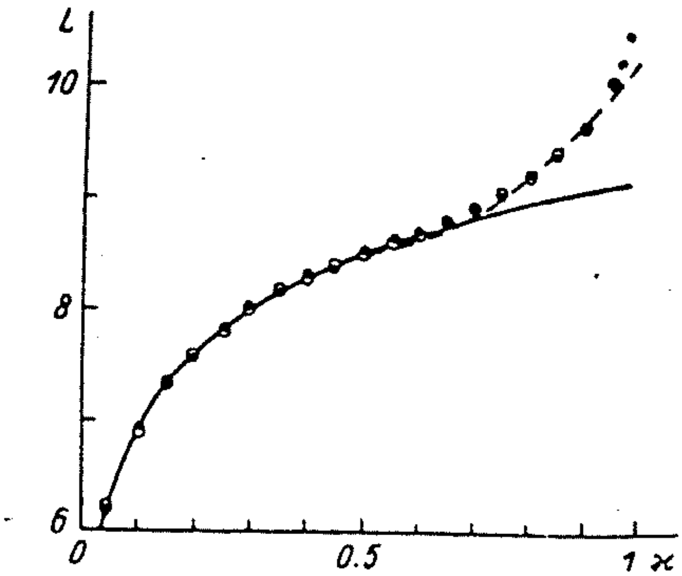

Figure 4: Maximum Lyapunov index vs cavity loss factor.

when $|\lambda| \to \infty$ (u and v being fixed). For this purpose, in accordance with the localization principle, it is necessary to multiply the integrand by unity, represented as a sum of two infinitely differentiable functions: $Q_1(S) + Q_2(S) \equiv 1; Q_1(S) \equiv 0$ for $S \leq \delta_1, Q_1(S) \equiv 1$ for $S > \delta_2$, where $0 < \delta_1 < \delta_2$. As a result, Φ_ν is transformed into a sum of two integrals. Considering the asymptotic properties of Bessel functions, one can show that the integral containing the function $Q_1(S)$ when $|\lambda| \to \infty$ has the order $O(|\lambda|^{-\infty})$. The asymptotic expansion for the other integral can be obtained by means of Erdèlyi's lemma; the leading term of this expansion is represented in the expression (A1). Note that when $k = 1$

$$\Phi_\nu(u,v) = \frac{1}{\nu\lambda} i^{\nu+1} J_\nu\left(\frac{\kappa u v}{2\lambda\nu}\right) \exp\left(-i\frac{v^2 + \kappa^2 u^2}{4\nu\lambda}\right), \tag{A2}$$

whence also follows the asymptotic estimate (A1).

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