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Expression for restricted partition function through Bernoulli polynomials

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Abstract Explicit expressions for restricted partition function $W(s,\mathbf{d}^m)$ and its quasiperiodic components $W_j(s,\mathbf{d}^m)$ (called *Sylvester waves*) for a set of positive integers $\mathbf{d}^m=\{d_1,d_2,\dots,d_m\}$ are derived. The formulas are represented in a form of a finite sum over Bernoulli polynomials of higher order with periodic coefficients.

Keywords Restricted partitions · Bernoulli polynomials of higher order

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1 Introduction

The problem of partitions of positive integers has long history started from the work of Euler who laid a foundation of the theory of partitions [1], introducing the idea of generating functions. Many prominent mathematicians contributed to the development of the theory using this approach.

J.J. Sylvester provided a new insight and made a remarkable progress in this field. He found [13, 14] the procedure for computation of a *restricted* partition function, which he called *denumerant* or *quotity*, and described symmetry properties of such functions. The restricted partition function $W(s,\mathbf{d}^m)\equiv W(s,\{d_1,d_2,\dots,d_m\})$ is a number of partitions of s into positive integers $\{d_1,d_2,\dots,d_m\}$, each not greater than s. The number $W(s,\mathbf{d}^m)$ is sometimes called *denumerant* and denoted D instead of W [10]. The generating function for $W(s,\mathbf{d}^m)$ has a form

$$F(t,\mathbf{d}^m)=\prod_{i=1}^{m}\frac{1}{1-t^{d_i}}=\sum_{s=0}^{\infty}W(s,\mathbf{d}^m)t^s \tag{1}$$

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where $W(s,\mathbf{d}^m)$ satisfies the basic recursive relation

$$W(s,\mathbf{d}^m) - W(s-d_m,\mathbf{d}^m) = W(s,\mathbf{d}^{m-1}). \tag{2}$$

Sylvester also proved the statement about splitting of the partition function into periodic and non-periodic parts and showed that the restricted partition function may be presented as a sum of "waves", which we call the *Sylvester waves*

$$W(s,\mathbf{d}^m) = \sum_{j=1} W_j(s,\mathbf{d}^m) \tag{3}$$

where summation runs over all divisors of the elements in the set $\mathbf{d}^m$. The wave $W_j(s,\mathbf{d}^m)$ is a quasipolynomial in s closely related to prime roots ρ_j of unity. Namely, Sylvester showed in [14] that the wave $W_j(s,\mathbf{d}^m)$ is a coefficient of t^{-1} in the series expansion in ascending powers of t of

$$F_j(s,t) = \sum_{\rho_j} \frac{\rho_j^{-s} e^{st}}{\prod_{k=1}^m (1-\rho_j^{d_k} e^{-d_k t})}. \tag{4}$$

The summation is made over all primitive roots of unity $\rho_j = \exp(2\pi i n/j)$ for n relatively prime to j (including unity) and smaller than j. This result is just a recipe for calculation of the partition function and it does not provide an explicit formula.

Using the Sylvester recipe we found in [12] an explicit formula for the Sylvester wave $W_j(s,\mathbf{d}^m)$ in a form of finite sum of the Bernoulli polynomials of higher order [2, 9] multiplied by a periodic function of integer period j. The periodic factor is expressed in terms of the Eulerian polynomials of higher order [4]. In this note we show that it is possible to express the partition function in terms of the Bernoulli polynomials only.

A special symbolic technique is developed in the theory of polynomials of higher order, which significantly simplifies computations performed with these polynomials. A short description of this technique required for better understanding of this paper is given in the Appendix.

2 Polynomial part of partition function and Bernoulli polynomials

Consider a polynomial part of the partition function corresponding to the wave $W_1(s,\mathbf{d}^m)$. It may be found as a residue of the generator

$$F_1(s,t) = \frac{e^{st}}{\prod_{i=1}^m (1-e^{-d_i t})}. \tag{5}$$

Recalling the generating function for the Bernoulli polynomials of higher order [2]:

$$\frac{e^{st} t^m \prod_{i=1}^m d_i}{\prod_{i=1}^m (e^{d_i t}-1)} = \sum_{n=0}^{\infty} B_n^{(m)}(s|\mathbf{d}^m) \frac{t^n}{n!}$$

and a transformation rule

$$B_n^{(m)}(s|-\mathbf{d}^m) = B_n^{(m)}\left(s + \sum_{i=1}^{m} d_i|\mathbf{d}^m\right),$$

we obtain the relation

$$\frac{e^{st}}{\prod_{i=1}^{m}(1-e^{-d_i t})} = \frac{1}{\pi_m}\sum_{n=0}^{\infty} B_n^{(m)}(s+s_m|\mathbf{d}^m)\frac{t^{n-m}}{n!} \tag{6}$$

where

$$s_m = \sum_{i=1}^{m} d_i, \qquad \pi_m = \prod_{i=1}^{m} d_i.$$

It is immediately seen from (6) that the coefficient of $1/t$ in (5) is given by the term with $n = m-1$

$$W_1(s,\mathbf{d}^m) = \frac{1}{(m-1)!\pi_m} B_{m-1}^{(m)}(s+s_m|\mathbf{d}^m). \tag{7}$$

The polynomial part also admits a symbolic form

$$W_1(s,\mathbf{d}^m) = \frac{1}{(m-1)!\pi_m}\left(s + s_m + \sum_{i=1}^{m} d_i{}^iB\right)^{m-1}$$

where after expansion powers r_i of ${}^iB$ are converted into orders of the Bernoulli numbers ${}^iB^{r_i} \Rightarrow B_{r_i}$. It is easy to recognize in (7) the explicit formula reported recently in [3], which was obtained by a straightforward computation of the complex residue of the generator (5).

Note that basic recursive relation for the Bernoulli polynomials [9]

$$B_n^{(m)}(s+d_m|\mathbf{d}^m) - B_n^{(m)}(s|\mathbf{d}^m) = nd_m B_{n-1}^{(m-1)}(s|\mathbf{d}^{m-1})$$

naturally leads to the basic recursive relation for the polynomial part of the partition function:

$$W_1(s,\mathbf{d}^m) - W_1(s-d_m,\mathbf{d}^m) = W_1(s,\mathbf{d}^{m-1}),$$

which coincides with (2). This indicates that the Bernoulli polynomials of higher order represent a natural basis for expansion of the partition function and its waves.

3 Sylvester waves and Eulerian polynomials

Frobenius [8] studied in great detail the so-called Eulerian polynomials $H_n(s,\rho)$ satisfying the generating function

$$\frac{(1-\rho)e^{st}}{e^t-\rho} = \sum_{n=0}^{\infty} H_n(s,\rho)\frac{t^n}{n!}, \quad \rho \neq 1,$$

which reduces to the definition of the Euler polynomials at fixed value of the parameter ρ

$$E_n(s) = H_n(s, -1).$$

The polynomials $H_n(\rho) \equiv H_n(0, \rho)$ satisfy the symbolic recursion ($H_0(\rho) = 1$)

$$\rho H_n(\rho) = (H(\rho) + 1)^n, \quad n > 0. \tag{8}$$

The generalization to higher orders is straightforward

$$\frac{e^{st} \prod_{i=1}^{m} (1 - \rho^{d_i})}{\prod_{i=1}^{m} (e^{d_i t} - \rho^{d_i})} = \sum_{n=0}^{\infty} H_n^{(m)}(s, \rho | \mathbf{d}^m) \frac{t^n}{n!}, \quad \rho^{d_i} \neq 1, \tag{9}$$

where the corresponding recursive relation for $H_n^{(m)}(s, \rho | \mathbf{d}^m)$ has the form

$$H_n^{(m)}(s + d_m, \rho | \mathbf{d}^m) - \rho^{d_m} H_n^{(m)}(s, \rho | \mathbf{d}^m) = (1 - \rho^{d_m}) H_n^{(m-1)}(s, \rho | \mathbf{d}^{m-1}).$$

The Eulerian polynomials of higher order $H_n^{(m)}(s, \rho | \mathbf{d}^m)$ introduced by Carlitz in [4] can be defined through the symbolic formula

$$H_n^{(m)}(s, \rho | \mathbf{d}^m) = \left(s + \sum_{i=1}^{m} d_i\, {}^i H(\rho^{d_i}) \right)^n$$

where $H_n(\rho)$ computed from the relation

$$\frac{1 - \rho}{e^t - \rho} = \sum_{n=0}^{\infty} H_n(\rho) \frac{t^n}{n!},$$

or using the recursion (8). Using the polynomials $H_n^{(m)}(s, \rho | \mathbf{d}^m)$ we can compute the Sylvester wave of arbitrary period.

In order to compute the Sylvester wave with period $j > 1$ we note that the summand in the expression (4) can be rewritten as a product

$$F_j(s, t) = \sum_{\rho_j} \frac{e^{st}}{\prod_{i=1}^{k_j} (1 - e^{-d_i t})} \times \frac{\rho_j^{-s}}{\prod_{i=k_j+1}^{m} (1 - \rho_j^{d_i} e^{-d_i t})} \tag{10}$$

where the elements in $\mathbf{d}^m$ are sorted in a way that j is a divisor for first k_j elements (we say that j has weight k_j), and the rest of the elements in the set are not divisible by j.

Consider a j-periodic Sylvester wave $W_j(s, \mathbf{d}^m)$, and rewrite the summand in (10) as double infinite sum using (6) and (9)

$$\frac{\rho_j^{-s}}{\pi_{k_j} \prod_{i=k_j+1}^{m} (1 - \rho_j^{d_i})} \sum_{n=0}^{\infty} B_n^{(k_j)}(s + s_{k_j} | \mathbf{d}^{k_j}) \frac{t^{n-k_j}}{n!}$$
$$\times \sum_{l=0}^{\infty} H_l^{(m-k_j)}(s_m - s_{k_j}, \rho_j | \mathbf{d}^{m-k_j}) \frac{t^l}{l!}. \tag{11}$$

The coefficient of $1/t$ in (11) is found for $n+l=k_j-1$, so that we obtain a finite sum:

$$W_j(s,\mathbf{d}^m)=\frac{1}{(k_j-1)!\pi_{k_j}}\sum_{\rho_j}\frac{\rho_j^{-s}}{\prod_{i=k_j+1}^m(1-\rho_j^{d_i})}\times\sum_{n=0}^{k_j-1}\binom{k_j-1}{n}B_n^{(k_j)}(s+s_{k_j}|\mathbf{d}^{k_j})H_{k_j-1-n}^{(m-k_j)}(s_m-s_{k_j},\rho_j|\mathbf{d}^{m-k_j}).$$

This expression may be rewritten as a symbolic power:

$$W_j(s,\mathbf{d}^m)=\frac{1}{(k_j-1)!\pi_{k_j}}\sum_{\rho_j}\frac{\rho_j^{-s}}{\prod_{i=k_j+1}^m(1-\rho_j^{d_i})}\times\left(s+s_m+\sum_{i=1}^{k_j}d_i{}^iB+\sum_{i=k_j+1}^{m}d_i{}^iH(\rho_j^{d_i})\right)^{k_j-1}, \tag{12}$$

which is equal to

$$W_j(s,\mathbf{d}^m)=\frac{1}{(k_j-1)!\pi_{k_j}}\sum_{n=0}^{k_j-1}\binom{k_j-1}{n}B_n^{(k_j)}(s+s_m|\mathbf{d}^{k_j})\times\sum_{\rho_j}\frac{\rho_j^{-s}}{\prod_{i=k_j+1}^m(1-\rho_j^{d_i})}H_{k_j-1-n}^{(m-k_j)}[\rho_j|\mathbf{d}^{m-k_j}] \tag{13}$$

where

$$H_n^{(m)}[\rho|\mathbf{d}^m]=H_n^{(m)}(0,\rho|\mathbf{d}^m)=\left[\sum_{i=1}^m d_i{}^iH(\rho^{d_i})\right]^n$$

are Eulerian numbers of higher order and it is assumed that

$$H_0^{(0)}[\rho|\emptyset]=1,\qquad H_n^{(0)}[\rho|\emptyset]=0,\quad n>0.$$

It should be noted that the presentation of the Sylvester wave as a finite sum of the Bernoulli polynomials with periodic coefficients (13) is not unique. The symbolic formula (12) can be cast into a sum of the Eulerian polynomials

$$W_j(s,\mathbf{d}^m)=\frac{1}{(k_j-1)!\pi_{k_j}}\sum_{n=0}^{k_j-1}\binom{k_j-1}{n}B_n^{(k_j)}[\mathbf{d}^{k_j}]\times\sum_{\rho_j}\frac{\rho_j^{-s}}{\prod_{i=k_j+1}^m(1-\rho_j^{d_i})}H_{k_j-1-n}^{(m-k_j)}(s+s_m,\rho_j|\mathbf{d}^{m-k_j}) \tag{14}$$

where

$$B_n^{(m)}[\mathbf{d}^m] = B_n^{(m)}(0|\mathbf{d}^m)$$

are the Bernoulli numbers of higher order.

It should be noted that the formula (13) requires summation over all primitive roots of unity, and though it is simpler than the Sylvester recipe using (4), it cannot be considered a completely explicit formula.

4 Reduction of Sylvester waves to Bernoulli polynomials

A relation between the Eulerian and Bernoulli numbers and polynomials of higher order established in [4] may be written as follows:

$$\frac{(m-1-n)!\pi_m \rho_j^{s_m}}{(k_j-1-n)!\pi_{k_j}\rho_j^{s_{k_j}}} \frac{1}{\prod_{i=k_j+1}^m (1-\rho_j^{d_i})} H_{k_j-1-n}^{(m-k_j)}[\rho_j|\mathbf{d}^{m-k_j}]$$
$$= j^{-(m-k_j)}\left(\prod_{i=k_j+1}^m \sum_{r_i=0}^{j-1} \rho_j^{-d_i r_i}\right) B_{m-1-n}^{(m-k_j)}\left(\sum_{i=k_j+1}^m d_i r_i | j\mathbf{d}^{m-k_j}\right).$$

Using this relation we convert the inner sum in (13) into

$$\sum_{\rho_j} \frac{\rho_j^{-s}}{\prod_{i=k_j+1}^m (1-\rho_j^{d_i})} H_{k_j-1-n}^{(m-k_j)}[\rho_j|\mathbf{d}^{m-k_j}]$$
$$= j^{-(m-k_j)} \frac{(k_j-1-n)!\pi_{k_j}}{(m-1-n)!\pi_m}$$
$$\times \sum_{\rho_j} \rho_j^{s_{k_j}-s_m-s}\left(\prod_{i=k_j+1}^m \sum_{r_i=0}^{j-1} \rho_j^{-d_i r_i}\right) B_{m-1-n}^{(m-k_j)}\left(\sum_{i=k_j+1}^m d_i r_i | j\mathbf{d}^{m-k_j}\right)$$
$$= j^{-(m-k_j)} \frac{(k_j-1-n)!\pi_{k_j}}{(m-1-n)!\pi_m}\left(\prod_{i=k_j+1}^m \sum_{r_i=0}^{j-1}\right) B_{m-1-n}^{(m-k_j)}\left(\sum_{i=k_j+1}^m d_i r_i | j\mathbf{d}^{m-k_j}\right)$$
$$\times \Psi_j\left(s + \sum_{i=k_j+1}^m d_i(r_i+1)\right). \tag{15}$$

Here $\Psi_j(s)$ denotes a *prime radical circulator* introduced in [5] (see also [6, 7])

$$\Psi_j(s) = \sum_{\rho_j} \rho_j^s.$$

For prime j it is given by

$$\Psi_j(s) = \begin{cases} \phi(j), & s = 0 \ (\mathrm{mod}\, j), \\ \mu(j), & s \neq 0 \ (\mathrm{mod}\, j) \end{cases} \tag{16}$$

where $\phi(j)$ and $\mu(j)$ denote Euler totient and Möbius functions. Considering j as a product of powers of distinct prime factors

$$j = \prod_k p_k^{\alpha_k},$$

one may easily check that for integer values of s

$$\Psi_j(s) = \prod_k p_k^{\alpha_k - 1} \Psi_{p_k}\left(\frac{s}{p_k^{\alpha_k - 1}}\right) \tag{17}$$

where $\Psi_k(s) = 0$ for non-integer values of s.

It is convenient to introduce a j*-modified* set of summands $\mathbf{d}_j^m$ defined as union of subset $\mathbf{d}^{k_j}$ of summands divisible by j and the remaining part $\mathbf{d}^{m-k_j}$ multiplied by j

$$\mathbf{d}_j^m = \mathbf{d}^{k_j} \cup j\mathbf{d}^{m-k_j},$$

so that $\mathbf{d}_j^m$ is divisible by j. Substitution of (15) into (13) with extension of the outer summation up to m produces the formula for computation of the j-periodic Sylvester wave:

$$W_j(s, \mathbf{d}^m) = \frac{1}{(m-1)!\pi_m j^{m-k_j}} \left(\prod_{i=k_j+1}^{m} \sum_{r_i=0}^{j-1} \right) B_{m-1}^{(m)} \left(s + s_m + \sum_{i=k_j+1}^{m} d_i r_i | \mathbf{d}_j^m \right)$$
$$\times \Psi_j \left(s + \sum_{i=k_j+1}^{m} d_i (r_i + 1) \right). \tag{18}$$

The derivation of (18) implies that all terms containing s to the power larger than $k_j - 1$ identically equal to zero. The polynomial part of the partition function $W_1(s, \mathbf{d}_j^m)$ for the j-modified set of summands reads:

$$W_1(s, \mathbf{d}_j^m) = \frac{1}{(m-1)!\pi_m j^{m-k_j}} B_{m-1}^{(m)} \left(s + s_m + (j-1) \sum_{i=k_j+1}^{m} d_i | \mathbf{d}_j^m \right).$$

This formula gives rise to the representation of the j-periodic Sylvester wave $W_j(s, \mathbf{d}^m)$ as the linear combination of the polynomial part of the j-modified set of summand $\mathbf{d}_j^m$ multiplied by the j-periodic functions Ψ_j:

$$W_j(s, \mathbf{d}^m) = \left(\prod_{i=k_j+1}^{m} \sum_{r_i=1}^{j} \right) W_1 \left(s + \sum_{i=k_j+1}^{m} d_i (r_i - j), \mathbf{d}_j^m \right) \Psi_j \left(s + \sum_{i=k_j+1}^{m} d_i (r_i - j) \right)$$

which is written also as

$$W_j(s, \mathbf{d}^m) = \left(\prod_{i=k_j+1}^{m} \sum_{r_i=0}^{j-1} \right) W_1 \left(s - \sum_{i=k_j+1}^{m} d_i r_i, \mathbf{d}_j^m \right) \Psi_j \left(s - \sum_{i=k_j+1}^{m} d_i r_i \right). \tag{19}$$

The last formula shows that each Sylvester wave is expressed as a linear superposition of the polynomial parts of the modified set of summands multiplied by the corresponding prime circulator. Thus, the formulas (16, 17, 19) with the Sylvester splitting formula (3) provide the explicit solution of the restricted partitions problem.

Appendix

The symbolic technique for manipulating sums with binomial coefficients by expanding polynomials and then replacing powers by subscripts was developed in nineteenth century by Blissard. It has been known as symbolic notation and the classical umbral calculus [11]. An example of this notation is also found in [2] in the section devoted to the Bernoulli polynomials $B_k(x)$.

The well-known formulas

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}, \qquad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}$$

are written symbolically as

$$B_n(x+y) = (B(x)+y)^n, \qquad B_n(x) = (B+x)^n.$$

After the expansion the exponents of $B(x)$ and B are converted into the orders of the Bernoulli polynomial and the Bernoulli number, respectively:

$$[B(x)]^k \Rightarrow B_k(x), \qquad B^k \Rightarrow B_k.$$

We use this notation in its extended version suggested in [9] in order to make derivation more clear and intelligible. Nörlund introduced the Bernoulli polynomials of higher order defined through the recursion

$$B_n^{(m)}(x|\mathbf{d}^m) = \sum_{k=0}^{n} \binom{n}{k} d_m^k B_k B_{n-k}^{(m-1)}(x|\mathbf{d}^{m-1}),$$

starting from $B_n^{(1)}(x|d_1) = d_1^n B_n(\frac{x}{d_1})$. In symbolic notation it takes the form

$$B_n^{(m)}(x) = \left(d_m B + B^{(m-1)}(x)\right)^n$$

and recursively reduces to more symmetric form

$$B_n^{(m)}(x|\mathbf{d}^m) = (x + d_1{}^1B + d_2{}^2B + \cdots + d_m{}^mB)^n = \left(x + \sum_{i=1}^{m} d_i{}^iB\right)^n$$

where each $[{}^iB]^k$ is converted into B_k.

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