Ramanujan J (2006) 11: 331–347
DOI 10.1007/s11139-006-8479-5

Restricted partition functions as Bernoulli and Eulerian polynomials of higher order

Boris Y. Rubinstein · Leonid G. Fel

Received: 12 May 2003 / Accepted: 3 July 2003
© Springer Science + Business Media, LLC 2006

Abstract Explicit expressions for restricted partition function $W(s, \mathbf{d}^m)$ and its quasiperiodic components $W_j(s, \mathbf{d}^m)$ (called *Sylvester waves*) for a set of positive integers $\mathbf{d}^m = \{d_1, d_2, \ldots, d_m\}$ are derived. The formulas are represented in a form of a finite sum over Bernoulli and Eulerian polynomials of higher order with periodic coefficients. A novel recursive relation for the Sylvester waves is established. Application to counting algebraically independent homogeneous polynomial invariants of finite groups is discussed.

Keywords Restricted partitions · Bernoulli polynomials of higher order · Eulerian polynomials of higher order · Recursive relation

2000 Mathematics Subject Classification Primary—11P81; Secondary—11B68, 11B37

1. Introduction

The problem of partitions of positive integers has long history started with the work of Euler who laid a foundation of the theory of partitions [1], by introducing the idea of generating functions. Many prominent mathematicians contributed to the development of the theory using the Euler idea.

J.J. Sylvester provided a new insight and made remarkable progress in this field. He found [15], [16] the procedure enabling to determine a *restricted* partition functions, and described symmetry properties of such functions. The restricted partition function

The research was supported in part (LGF) by the Gileadi Fellowship program of the Ministry of Absorption of the State of Israel.

B. Y. Rubinstein (✉)
Department of Mathematics, University of California, Davis, One Shields Dr., Davis, CA 95616, U.S.A.
e-mail: boris@math.ucdavis.edu

L. G. Fel
Department of Civil and Environmental Engineering, Technion, Haifa 32000, Israel

$W(s, \mathbf{d}^m) \equiv W(s, \{d_1, d_2, \dots, d_m\})$ is the number of partitions of s into positive integers $\{d_1, d_2, \dots, d_m\}$, each not greater than s. The generating function for $W(s, \mathbf{d}^m)$ has the form

$$F(t, \mathbf{d}^m) = \prod_{i=1}^{m} \frac{1}{1 - t^{d_i}} = \sum_{s=0}^{\infty} W(s, \mathbf{d}^m)\, t^s \ , \tag{1}$$

where $W(s, \mathbf{d}^m)$ satisfies the basic recursive relation

$$W(s, \mathbf{d}^m) - W(s - d_m, \mathbf{d}^m) = W(s, \mathbf{d}^{m-1}) \ . \tag{2}$$

Sylvester also proved the statement about splitting of the partition function into periodic and non-periodic parts and showed that the restricted partition function may be presented as a sum of "waves", which we call the *Sylvester waves*

$$W(s, \mathbf{d}^m) = \sum_{j=1} W_j(s, \mathbf{d}^m) \ , \tag{3}$$

where summation runs over all distinct factors in the set $\mathbf{d}^m$. The wave $W_j(s, \mathbf{d}^m)$ is a quasipolynomial in s closely related to primitive roots ρ_j of unity. Namely, Sylvester showed in [16] that the wave $W_j(s, \mathbf{d}^m)$ is a coefficient of t^{-1} in the series expansion in ascending powers of t of

$$F_j(s, t) = \sum_{\rho_j} \frac{\rho_j^{-s} e^{st}}{\prod_{k=1}^{m} (1 - \rho_j^{d_k} e^{-d_k t})} \ . \tag{4}$$

The summation is made over all primitive roots of unity $\rho_j = \exp(2\pi i n/j)$ for n relatively prime to j (including unity) and smaller than j. This result is just a recipe for calculation of the partition function and it does not provide any explicit formula.

Using the Sylvester recipe we find an explicit formula for the Sylvester wave $W_j(s, \mathbf{d}^m)$ in a form of finite sum of the Bernoulli polynomials of higher order [2], [7] multiplied by a periodic function of integer period j. The periodic factor is expressed through the generalized Eulerian polynomials of higher order [5].

A special symbolic technique is developed in the theory of polynomials of higher order, which significantly simplifies computations performed with these polynomials. A short description of this technique required for better understanding of this paper is given in Appendix A.

2. Sylvester wave $W_1(s, \mathbf{d}^m)$ and Bernoulli polynomials of higher order

Consider a polynomial part of the partition function corresponding to the wave $W_1(s, \mathbf{d}^m)$. It may be found as a residue of the generator

$$F_1(s, t) = \frac{e^{st}}{\prod_{i=1}^{m} (1 - e^{-d_i t})} \ . \tag{5}$$

Recalling the generating function for the Bernoulli polynomials of higher order [2]:

$$\frac{e^{st}t^m \prod_{i=1}^m d_i}{\prod_{i=1}^m (e^{d_i t}-1)} = \sum_{n=0}^{\infty} B_n^{(m)}(s|\mathbf{d}^m)\frac{t^n}{n!}, \tag{6}$$

and a transformation rule

$$B_n^{(m)}(s|-\mathbf{d}^m) = B_n^{(m)}\left(s+\sum_{i=1}^m d_i|\mathbf{d}^m\right),$$

we obtain the relation

$$\frac{e^{st}}{\prod_{i=1}^m (1-e^{-d_i t})} = \frac{1}{\pi_m}\sum_{n=0}^{\infty} B_n^{(m)}(s+s_m|\mathbf{d}^m)\frac{t^{n-m}}{n!}, \tag{7}$$

where

$$s_m = \sum_{i=1}^m d_i, \quad \pi_m = \prod_{i=1}^m d_i .$$

It is immediately seen from (7) that the coefficient of $1/t$ in (5) is given by the term with $n=m-1$

$$W_1(s,\mathbf{d}^m) = \frac{1}{(m-1)!\,\pi_m} B_{m-1}^{(m)}(s+s_m|\mathbf{d}^m)\ . \tag{8}$$

The polynomial part also admits a symbolic form frequently used in theory of higher order polynomials

$$W_1(s,\mathbf{d}^m) = \frac{1}{(m-1)!\,\pi_m}\left(s+s_m+\sum_{i=1}^m d_i\ {}^iB\right)^{m-1}, \tag{9}$$

where after expansion powers r_i of ${}^iB$ are converted into orders of the Bernoulli numbers

$${}^iB^{r_i} \Rightarrow B_{r_i}\ . \tag{10}$$

It is easy to recognize in (8) the explicit formula reported recently in [3], which was obtained by a straightforward computation of the complex residue of the generator (5).

Note that basic recursive relation for the Bernoulli polynomials [7]

$$B_n^{(m)}(s+d_m|\mathbf{d}^m) - B_n^{(m)}(s|\mathbf{d}^m) = nd_m B_{n-1}^{(m-1)}(s|\mathbf{d}^{m-1}) \tag{11}$$

naturally leads to the basic recursive relation for the polynomial part of the partition function:

$$W_1(s,\mathbf{d}^m) - W_1(s-d_m,\mathbf{d}^m) = W_1(s,\mathbf{d}^{m-1})\ , \tag{12}$$

which coincides with (2). This indicates that the Bernoulli polynomials of higher order represent a natural basis for expansion of the partition function and its waves.

3. Sylvester wave $W_2(s, \mathbf{d}^m)$ and Euler polynomials of higher order

In order to compute the Sylvester wave with period $j > 1$ we note that the expression (4) can be rewritten as a product

$$F_j(s,t) = \sum_{\rho_j} \frac{e^{st}}{\prod_{i=1}^{\omega_j}(1-e^{-d_i t})} \times \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i} e^{-d_i t})}, \tag{13}$$

where the elements in $\mathbf{d}^m$ are sorted in a way that j is a divisor for first ω_j elements (we say that j has weight ω_j), and the rest of the elements in the set are not divisible by j.

Then a 2-periodic Sylvester wave $W_2(s, \mathbf{d}^m)$ is a residue of the generator

$$F_2(s,t) = \frac{e^{st}}{\prod_{i=1}^{\omega_2}(1-e^{-d_i t})} \times \frac{(-1)^s}{\prod_{i=\omega_2+1}^{m}(1+e^{-d_i t})}, \tag{14}$$

where first ω_2 integers d_i are even, and the summation is omitted being trivially restricted to the only value $\rho_2 = -1$. Recalling the generating function for the Euler polynomials of higher order [2], [7] and corresponding recursive relation

$$\frac{2^m e^{st}}{\prod_{i=1}^{m}(e^{d_i t}+1)} = \sum_{n=0}^{\infty} E_n^{(m)}(s|\mathbf{d}^m)\frac{t^n}{n!}, \tag{15}$$
$$E_n^{(m)}(s+d_m|\mathbf{d}^m) + E_n^{(m)}(s|\mathbf{d}^m) = 2E_n^{(m-1)}(s|\mathbf{d}^{m-1}),$$

we may rewrite (14) as double infinite sum

$$\frac{(-1)^s}{2^{m-\omega_2}\pi_{\omega_2}} \sum_{n=0}^{\infty} B_n^{(\omega_2)}(s+s_{\omega_2}|\mathbf{d}^{\omega_2})\frac{t^{n-\omega_2}}{n!} \sum_{l=0}^{\infty} E_l^{(m-\omega_2)}(s_m - s_{\omega_2}|\mathbf{d}^{m-\omega_2})\frac{t^l}{l!}\ . \tag{16}$$

The coefficient of $1/t$ in the above series is found for $n + l = \omega_2 - 1$, so that we end up with a finite sum:

$$W_2(s,\mathbf{d}^m) = \frac{(-1)^s}{(\omega_2-1)!\,2^{m-\omega_2}\pi_{\omega_2}} \sum_{n=0}^{\omega_2-1} \binom{\omega_2-1}{n} B_n^{(\omega_2)}(s+s_{\omega_2}|\mathbf{d}^{\omega_2}) E_{\omega_2-1-n}^{(m-\omega_2)}(s_m - s_{\omega_2}|\mathbf{d}^{m-\omega_2}). \tag{17}$$

This expression may be rewritten as a symbolic power similar to (9):

$$W_2(s,\mathbf{d}^m) = \frac{(-1)^s}{(\omega_2-1)!\,2^{m-\omega_2}\pi_{\omega_2}} \left(s + s_m + \sum_{i=1}^{\omega_2} d_i\ {}^iB + \sum_{i=\omega_2+1}^{m} d_i\ {}^iE(0)\right)^{\omega_2-1}, \tag{18}$$

where the rule for the Euler polynomials at zero $E_n(0)$ similar to (10) is applied. It is easy to rewrite formula (18) in a form

$$W_2(s, \mathbf{d}^m) = \frac{(-1)^s}{(\omega_2 - 1)!\; 2^{m-\omega_2}\pi_{\omega_2}} \sum_{n=0}^{\omega_2-1} \binom{\omega_2 - 1}{n} B_n^{(\omega_2)}(s + s_m|\mathbf{d}^{\omega_2}) E_{\omega_2-1-n}^{(m-\omega_2)}(0|\mathbf{d}^{m-\omega_2}), \tag{19}$$

where $E_n^{(m)}(0|\mathbf{d}^m)$ denote the Euler polynomials of higher orders computed at zero as follows:

$$E_n^{(m)}(0|\mathbf{d}^m) = \left[\sum_{i=1}^{m} d_i \; {}^iE(0)\right]^n . \tag{20}$$

The formula (19) shows that the wave $W_2(s, \mathbf{d}^m)$ can be written as an expansion over the Bernoulli polynomials of higher order with constant coefficients, multiplied by a 2-periodic function $(-1)^s$.

4. Sylvester waves $W_j(s, \mathbf{d}^m)$ ($j > 2$) and Eulerian polynomials of higher order

Frobenius [9] studied in great detail the polynomials $H_n(s, \rho)$ satisfying the generating function

$$\frac{(1-\rho)e^{st}}{e^t - \rho} = \sum_{n=0}^{\infty} H_n(s, \rho)\frac{t^n}{n!}, \quad (\rho \neq 1), \tag{21}$$

which reduces to definition of the Eulerian polynomials at fixed value of the parameter ρ

$$E_n(s) = H_n(s, -1).$$

The polynomials $H_n(\rho) \equiv H_n(0, \rho)$ satisfy the symbolic recursion ($H_0(\rho) = 1$)

$$\rho H_n(\rho) = (H(\rho) + 1)^n, \quad n > 0. \tag{22}$$

The generalization of (15) is straightforward

$$\frac{e^{st}\prod_{i=1}^{m}(1-\rho^{d_i})}{\prod_{i=1}^{m}(e^{d_i t} - \rho^{d_i})} = \sum_{n=0}^{\infty} H_n^{(m)}(s, \rho|\mathbf{d}^m)\frac{t^n}{n!}, \quad (\rho^{d_i} \neq 1), \tag{23}$$

where the corresponding recursive relation for $H_n^{(m)}(s, \rho|\mathbf{d}^m)$ has the form

$$H_n^{(m)}(s + d_m, \rho|\mathbf{d}^m) - \rho^{d_m} H_n^{(m)}(s, \rho|\mathbf{d}^m) = (1 - \rho^{d_m}) H_n^{(m-1)}(s, \rho|\mathbf{d}^{m-1}) . \tag{24}$$

The *generalized Eulerian polynomials of higher order* $H_n^{(m)}(s,\rho|\mathbf{d}^m)$ introduced by L. Carlitz in [5] can be defined through the symbolic formula

$$H_n^{(m)}(s,\rho|\mathbf{d}^m) = \left(s + \sum_{i=1}^{m} d_i\ {}^iH(\rho^{d_i})\right)^n , \tag{25}$$

where $H_n(\rho)$ computed from the relation

$$\frac{1-\rho}{e^t-\rho} = \sum_{n=0}^{\infty} H_n(\rho)\frac{t^n}{n!},$$

or using the recursion (22). Using the polynomials $H_n^{(m)}(s,\rho|\mathbf{d}^m)$ we can compute Sylvester wave of arbitrary period.

Consider a j-periodic Sylvester wave $W_j(s,\mathbf{d}^m)$, and rewrite the summand in (13) as double infinite sum

$$\frac{\rho_j^{-s}}{\pi_{\omega_j}\ \prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})} \sum_{n=0}^{\infty} B_n^{(\omega_j)}(s+s_{\omega_j}|\mathbf{d}^{\omega_j})\frac{t^{n-\omega_j}}{n!}$$
$$\times \sum_{l=0}^{\infty} H_l^{(m-\omega_j)}(s_m - s_{\omega_j},\rho_j|\mathbf{d}^{m-\omega_j})\frac{t^l}{l!}. \tag{26}$$

The coefficient of $1/t$ in the above series is found for $n+l=\omega_j-1$, so that we have a finite sum:

$$W_j(s,\mathbf{d}^m) = \frac{1}{(\omega_j-1)!\ \pi_{\omega_j}} \sum_{\rho_j} \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})}$$
$$\times \sum_{n=0}^{\omega_j-1} \binom{\omega_j-1}{n} B_n^{(\omega_j)}(s+s_{\omega_j}|\mathbf{d}^{\omega_j}) H_{\omega_j-1-n}^{(m-\omega_j)}(s_m-s_{\omega_j},\rho_j|\mathbf{d}^{m-\omega_j}) . \tag{27}$$

This expression may be rewritten as a symbolic power similar to (18):

$$W_j(s,\mathbf{d}^m) = \frac{1}{(\omega_j-1)!\ \pi_{\omega_j}} \sum_{\rho_j} \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})}$$
$$\times \left(s + s_m + \sum_{i=1}^{\omega_j} d_i\ {}^iB + \sum_{i=\omega_j+1}^{m} d_i\ {}^iH(\rho_j^{d_i})\right)^{\omega_j-1} , \tag{28}$$

which is equal to

$$W_j(s,\mathbf{d}^m) = \frac{1}{(\omega_j-1)!\ \pi_{\omega_j}} \sum_{n=0}^{\omega_j-1} \binom{\omega_j-1}{n} B_n^{(\omega_j)}(s+s_m|\mathbf{d}^{\omega_j})$$
$$\times \sum_{\rho_j} \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})} H_{\omega_j-1-n}^{(m-\omega_j)}[\rho_j|\mathbf{d}^{m-\omega_j}], \tag{29}$$

where

$$H_n^{(m)}[\rho|\mathbf{d}^m] = H_n^{(m)}(0,\rho|\mathbf{d}^m) = \left[\sum_{i=1}^m d_i\ {}^i H(\rho^{d_i})\right]^n, \tag{30}$$

are generalized Eulerian numbers of higher order and it is assumed that

$$H_0^{(0)}[\rho|\emptyset] = 1,\ \ H_n^{(0)}[\rho|\emptyset] = 0,\ \ n > 0.$$

It should be underlined that the presentation of the Sylvester wave as a finite sum of the Bernoulli polynomials with periodic coefficients (29) is not unique. The symbolic formula (28) can be cast into a sum of the generalized Eulerian polynomials

$$\begin{aligned} W_j(s,\mathbf{d}^m) = {} & \frac{1}{(\omega_j-1)!\ \pi_{\omega_j}} \sum_{n=0}^{\omega_j-1} \binom{\omega_j-1}{n} B_n^{(\omega_j)}[\mathbf{d}^{\omega_j}] \\ & \times \sum_{\rho_j} \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^m (1-\rho_j^{d_i})} H_{\omega_j-1-n}^{(m-\omega_j)}(s+s_m,\rho_j|\mathbf{d}^{m-\omega_j}), \end{aligned} \tag{31}$$

where

$$B_n^{(m)}[\mathbf{d}^m] = B_n^{(m)}(0|\mathbf{d}^m)$$

are the Bernoulli numbers of higher order.

Substitution of the expression (29) into the expansion (3) immediately produces the partition function $W(s,\mathbf{d}^m)$ as finite sum of the Bernoulli polynomials of higher order multiplied by periodic functions with period equal to the least common multiple of the elements in $\mathbf{d}^m$

$$\begin{aligned} W(s,\mathbf{d}^m) = {} & \sum_j \frac{1}{(\omega_j-1)!\ \pi_{\omega_j}} \sum_{n=0}^{\omega_j-1} \binom{\omega_j-1}{n} B_n^{(\omega_j)}(s+s_m|\mathbf{d}^{\omega_j}) \\ & \times \sum_{\rho_j} \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^m (1-\rho_j^{d_i})} H_{\omega_j-1-n}^{(m-\omega_j)}[\rho_j|\mathbf{d}^{m-\omega_j}]. \end{aligned} \tag{32}$$

The partition function $W(s,\mathbf{d}^m)$ has several interesting properties. Analysis of the generating function (1) shows that the partition function is a homogeneous function of zero order with respect to all its arguments, i.e.,

$$W(ks,k\mathbf{d}^m) = W(s,\mathbf{d}^m). \tag{33}$$

This property appears very useful for computation of the partition function in case when the elements d_i have a common factor k, then

$$W(s,k\mathbf{d}^m) = W\left(\frac{s}{k},\mathbf{d}^m\right). \tag{34}$$

The case of m identical elements $\mathbf{p}^m = \{p, \dots, p\}$ appears to be the simplest and is reduced to the known formula for Catalan partitions [6]: *the Diophantine equation* $x_1 + x_2 + \cdots + x_m = s$ *has* $\binom{s+m-1}{s}$ *sets of non-negative solutions.*

Using (34) for s divisible by p we arrive at

$$W(s, \mathbf{p}^m) = W\left(\frac{s}{p}, \mathbf{1}^m\right) = W_1\left(\frac{s}{p}, \mathbf{1}^m\right) = \frac{B_{m-1}^{(m)}(s/p + m|\mathbf{1}^m)}{(m-1)!}.$$

A straightforward computation shows that

$$B_{m-1}^{(m)}(s + m|\mathbf{1}^m) = \prod_{k=1}^{m-1}(s+k) = \frac{(s+m-1)!}{s!},$$

so that

$$W(s, \mathbf{p}^m) = \begin{cases} \prod_{k=1}^{m-1}\left(1 + \dfrac{s}{kp}\right), & s = 0 \pmod{p}, \\ 0, & s \neq 0 \pmod{p}. \end{cases} \tag{35}$$

At the end of this Section we consider a special case of the tuple $\{p_1, p_2, \dots p_m\}$ of primes p_j which leads to essential simplification of the formula (32). The first Sylvester wave W_1 is given by (8) while all higher waves arising are purely periodic

$$W_{p_i}(s; \{p_1, p_2, \dots, p_m\}) = \frac{1}{p_i}\sum_{k=1}^{p_i-1}\frac{\rho_{p_i}^{-ks}}{\prod_{j\neq i}^{m}(1 - \rho_{p_i}^{kp_j})}\ . \tag{36}$$

The further simplification $m = 2,\ s = ap_1p_2$ makes it possible to verify the partition identity

$$W(ap_1p_2, \{p_1, p_2\}) = a + 1, \tag{37}$$

which follows from the recursion relation (2) for the restricted partition function and its definition

$$\begin{aligned} W(ap_1p_2, \{p_1, p_2\}) - W(ap_1p_2 - p_1, \{p_1, p_2\}) &= W(ap_1p_2, \{p_2\}), \\ W(ap_1p_2, \{p_2\}) &= 1, \\ W(ap_1p_2 - p_1, \{p_1, p_2\}) = W((a-l)p_1p_2 + (lp_2 - 1)p_1, \{p_1, p_2\}) &= a, \end{aligned}$$

where a solutions of the Diophantine equation $p_1X + p_2Y = (a-l)p_1p_2 + (lp_2 - 1)p_1$ correspond to $l = 1, \dots, a$. The relation (37) has an important geometrical interpretation, namely, a line $p_1X + p_2Y = ap_1p_2$ in the XY plane passes exactly through $a + 1$ points with non-negative integer coordinates.

The verification of (37) is straightforward (see Appendix B for details):

$$\begin{aligned} &W_1(ap_1p_2, \{p_1, p_2\}) = a + \frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right), \\ &W_{p_1}(ap_1p_2, \{p_1, p_2\}) = \frac{1}{2} - \frac{1}{2p_1}, \quad W_{p_2}(ap_1p_2, \{p_1, p_2\}) = \frac{1}{2} - \frac{1}{2p_2}, \end{aligned} \tag{38}$$

which produces the required result.

A generalization of (37) is possible using the explicit form of the partition function

$$W(s,\{p_1,p_2\}) = \frac{1}{p_1p_2}\left(s+\frac{p_1+p_2}{2}\right)+\frac{1}{p_1}\sum_{\rho_{p_1}}\frac{\rho_{p_1}^{-s}}{1-\rho_{p_1}^{p_2}}+\frac{1}{p_2}\sum_{\rho_{p_2}}\frac{\rho_{p_2}^{-s}}{1-\rho_{p_2}^{p_1}}. \quad (39)$$

Setting here $s = ap_1p_2+b,\ 0\le b<p_1p_2$ and noting that the value of two last terms in (39) don't depend on the integer a, one can easily see that

$$W(ap_1p_2+b,\{p_1,p_2\}) = a + W(b,\{p_1,p_2\}), \quad (40)$$

which reduces the procedure to computation of the first p_1p_2 values of $W(s,\{p_1,p_2\})$. Recalling that $W(0,\{p_1,p_2\})=1$ we immediately recover (37) as a particular case of (40).

5. Recursive relation for Sylvester waves

In this Section we prove that the recursive relation similar to (2) holds not only for the entire partition function $W(s,\mathbf{d}^m)$ and its polynomial part $W_1(s,\mathbf{d}^m)$ but also for each Sylvester wave

$$W_j(s,\mathbf{d}^m) - W_j(s-d_m,\mathbf{d}^m) = W_j(s,\mathbf{d}^{m-1})\ . \quad (41)$$

When j is not a divisor of d_m, the weight ω_j doesn't change in transition from $\mathbf{d}^{m-1}$ to $\mathbf{d}^m$. Denoting for brevity

$$A(s) = s + s_{m-1} + \sum_{i=1}^{\omega_j} d_i\ {}^{i}B + \sum_{i=\omega_j+1}^{m-1} d_i\ {}^{i}H\big(\rho_j^{d_i}\big),\quad B_{\omega_j} = \frac{1}{(\omega_j-1)!\ \pi_{\omega_j}},$$

we have

$$\begin{aligned} W_j(s,\mathbf{d}^m) &= B_{\omega_j}\sum_{\rho_j}\frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})}\big(A(s)+d_m\big[1+H\big(\rho_j^{d_m}\big)\big]\big)^{\omega_j-1} \\ &= B_{\omega_j}\sum_{\rho_j}\frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})}\sum_{l=0}^{\omega_j-1}\binom{\omega_j-1}{l}A^{\omega_j-1-l}(s)d_m^l\big[1+H\big(\rho_j^{d_m}\big)\big]^l\ . \end{aligned}$$

Now using (22) we have

$$\begin{aligned} &W_j(s,\mathbf{d}^m) \\ &= B_{\omega_j}\sum_{\rho_j}\frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})}\left\{A^{\omega_j-1}(s)+\rho_j^{d_m}\sum_{l=1}^{\omega_j-1}\binom{\omega_j-1}{l}A^{\omega_j-1-l}(s)d_m^l H_l(\rho_j^{d_m})\right\} \end{aligned}$$

$$
\begin{aligned}
&= B_{\omega_j} \sum_{\rho_j} \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})} \left\{(1-\rho_j^{d_m})A^{\omega_j-1}(s) + \rho_j^{d_m}\left(A(s)+d_m H(\rho_j^{d_m})\right)^{\omega_j-1}\right\} \\
&= B_{\omega_j} \sum_{\rho_j} \frac{\rho_j^{-(s-d_m)}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})}\left(A(s)+d_m H(\rho_j^{d_m})\right)^{\omega_j-1} + B_{\omega_j}\sum_{\rho_j}\frac{\rho_j^{-s}A^{\omega_j-1}(s)}{\prod_{i=\omega_j+1}^{m-1}(1-\rho_j^{d_i})} \\
&= W_j(s-d_m, \mathbf{d}^m) + W_j(s, \mathbf{d}^{m-1})\ . \qquad (42)
\end{aligned}
$$

In case of j being divisor of d_m the weight of j for the set $\mathbf{d}^{m-1}$ is equal to $\omega_j - 1$, and we have

$$
\begin{aligned}
W_j(s, \mathbf{d}^{m-1}) = \frac{(\omega_j-1)d_m}{(\omega_j-1)!\,\pi_{\omega_j}} \sum_{\rho_j} \frac{\rho_j^{-s}}{\prod_{i=\omega_j}^{m-1}(1-\rho_j^{d_i})} \\
\times \left(s + s_{m-1} + \sum_{i=1}^{\omega_j-1} d_i\ {}^iB + \sum_{i=\omega_j}^{m-1} d_i\ {}^iH\left(\rho_j^{d_i}\right)\right)^{\omega_j-2}. \qquad (43)
\end{aligned}
$$

Denoting

$$
A(s) = s + s_{m-1} + \sum_{i=1}^{\omega_j-1} d_i\ {}^iB + \sum_{i=\omega_j}^{m-1} d_i\ {}^iH\left(\rho_j^{d_i}\right), \quad D(s,\rho_j) = \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})},
$$

and using the symbolic formula for the Bernoulli numbers [7]

$$
(B+1)^n = B^n = B_n \ \ (n \neq 1),
$$

we obtain

$$
\begin{aligned}
W_j(s, \mathbf{d}^m) &= B_{\omega_j} \sum_{\rho_j} D(s,\rho_j)[A(s) + d_m(B+1)]^{\omega_j-1} \\
&= B_{\omega_j} \sum_{\rho_j} D(s,\rho_j) \sum_{l=0}^{\omega_j-1} \binom{\omega_j-1}{l} A^{\omega_j-1-l}(s) d_m^l (B+1)^l \qquad (44) \\
&= B_{\omega_j} \sum_{\rho_j} D(s,\rho_j)[A(s)+d_m B]^{\omega_j-1} + B_{\omega_j} d_m(\omega_j-1)\sum_{\rho_j} D(s,\rho_j)A^{\omega_j-2}(s) \\
&= W_j(s-d_m, \mathbf{d}^m) + W_j(s, \mathbf{d}^{m-1}),
\end{aligned}
$$

which completes the proof.

6. Partition function $W(s, \{\overline{m}\})$ for a set of natural numbers

Sylvester waves for a set of consecutive natural numbers $\{1, 2, \ldots, m\} = \{\overline{m}\}$ was under special consideration in [8]. An importance of this case based on its relation to the invariants of symmetric group S_m (see next Section) and, second, $W(s, \{\overline{m}\})$ form a natural basis to utilize the partition functions for every subsets of $\{1, 2, \ldots, m\}$. This case is also important due to the famous Rademacher formula [11] for *unrestricted partition function* $W(s, \{\overline{s}\})$, but the latter already belongs to the analytical number theory.

The representation for $W(s, \{\overline{m}\})$ in terms of higher Bernoulli polynomials comes when we put into (32)

$$\omega_j = \left[\frac{m}{j}\right], \quad \pi_{\omega_j} = \omega_j!\ j^{\omega_j}, \quad s_{\omega_j} = \frac{\omega_j(\omega_j+1)}{2}, \tag{45}$$

where $[x]$ denotes integer part of x. The partition function in this case reads

$$W(s, \{\overline{m}\}) = \sum_{j=1}^{m} \frac{j^{-\omega_j}}{(\omega_j-1)!\ \omega_j!} \sum_{n=0}^{\omega_j-1} \binom{\omega_j-1}{n} B_n^{(\omega_j)}\left(s + \frac{m(m+1)}{2}|\mathbf{d}^{\omega_j}\right)$$
$$\times \sum_{\rho_j} \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1-\rho_j^{d_i})} H_{\omega_j-1-n}^{(m-\omega_j)}[\rho_j|\mathbf{d}^{m-\omega_j}], \tag{46}$$

where $\mathbf{d}^{\omega_j} = j\{\overline{\omega}\}$, so that for each j we have elements divisible by j at first ω_j positions. The expression for the Sylvester wave of the maximal period m looks particularly simple

$$W_m(s, \{\overline{m}\}) = \frac{1}{m^2}\sum_{\rho_m} \rho_m^{-s}\ . \tag{47}$$

Straightforward calculations show that the expression (46) produces exactly the same formulas for $m = 1, 2, \ldots, 12$ which were obtained in [8].

It needs to be noted that typically the argument s in all formulas derived above is assumed to have integer values, but it is obvious that all results can be extended to real values of s, though such extension is not unique. Continuous values of the argument provide a convenient way to analyze the behavior of the partition function and its waves. In this work we choose the natural extension scheme based on the trigonometric functions

$$\rho_j^s = e^{2\pi i n s/j} = \cos\frac{2\pi n s}{j} + i\sin\frac{2\pi n s}{j}.$$

We finish this Section with a brief discussion of a phenomenon better observed in graphics of $W(s, \{\overline{m}\})$ with large m rather than explicit expressions (see formulas (52) and Figures of restricted partition functions in [8]).

In the range $\left[-\frac{m(m+1)}{2}, 0\right]$ where $W(s, \{\overline{m}\})$ has all its zeroes, one can easily assume the existence of a function $\widetilde{W}(s, \{\overline{m}\})$ which envelopes $W(s, \{\overline{m}\})$ or approximates it in some sense. The decomposition of $W(s, \{\overline{m}\})$ into the Sylvester waves shows that this role may be assigned to the wave $W_1(s, \{\overline{m}\})$. The Figs 1 and 2 show that $W_1(s, \{\overline{21}\})$ serves as a good approximant for $W(s, \{\overline{21}\})$ in this range as well as for large s.

7. Application to invariants of finite groups

The restricted partition function $W(s, \mathbf{d}^m)$ has a strong relationship to the invariants of finite reflection groups G acting on the vector space V over the field of complex numbers. If

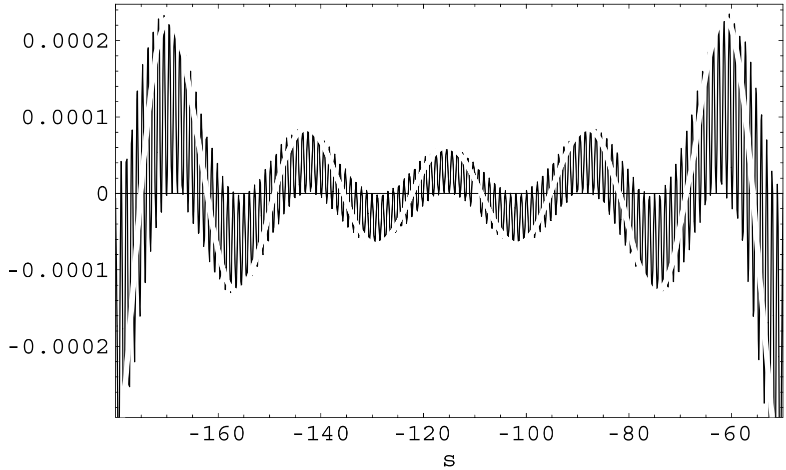

Fig. 1 Plots of the partition function $W(s, \{\overline{21}\})$ (*black curve*) and its first Sylvester wave $W_1(s, \{\overline{21}\})$ (*white curve*) showing that the polynomial part provides an important information about the partition function behavior

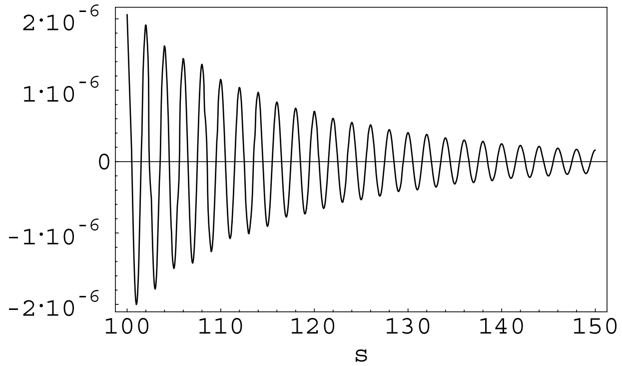

Fig. 2 Plot of the normalized difference $[W(s, \{\overline{21}\})/W_1(s, \{\overline{21}\}) - 1]$ showing that the polynomial part $W_1(s, \{\overline{21}\})$ at large values of the argument s gives a very accurate approximation to the partition function $W(s, \{\overline{21}\})$

$M^G(t)$ is a Molien function of the finite group, d_r and m are degrees and a number of the basic homogeneous invariants respectively, then its series expansion in t gives a number $P(s, G)$ of algebraically independent invariants of the degree s. The set of natural numbers $\{\overline{m}\}$ corresponds to the symmetric group S_m : $W(s, \{\overline{m}\}) = P(s, S_m)$. The list of $P(s, G)$ for all indecomposable reflections groups G acting over the field of real numbers and known as *Coxeter groups* is presented in [8]. It is easy to extend these formulas over indecomposable pseudoreflections groups acting over the field of complex numbers using the list of 37 groups given by Shepard and Todd [13]. In this Section we extend the results of Section 4 to all finite groups.

First, we recall an algebraic setup of the problem. The fundamental problem of the invariant theory consists in determination of an algebra R^G of invariants. Its solution is given by the Noether theorem [4]: R^G is generated by a polynomial $\vartheta_k(x_j)$ as an algebra due to action of finite group $G \subset GL(V^q)$ on the q-dimensional vector space $V^q(x_j)$ over the field of complex numbers by not more than $\binom{|G|+q}{q}$ homogeneous invariants, of degrees not exceeding the order $|G|$ of group

$$k \le \binom{|G|+q}{q}, \quad j \le \dim V^q = q, \quad \deg \vartheta_k(x_j) \le |G| \,. \tag{48}$$

To enumerate the invariants explicitly, it is convenient to classify them by their degrees (as polynomials). A classical theorem of Molien [4] gives an explicit expression for a number $P(s, G)$ of all homogeneous invariants of degree s

$$M^G(t) = \frac{1}{|G|}\sum_{l=1}^{|G|}\frac{\widetilde{\chi}(\widehat{g}_l)}{\det(\hat{I} - t\,\widehat{g}_l)} = \sum_{s=0}^{\infty} P(s, G)t^s, \quad P(0, G) = 1, \tag{49}$$

where $\widehat{g}_l$ are non-singular $(n \times n)$-permutation matrices with entries, which form the regular representation of G, $\widehat{I}$ is the identity matrix and $\widetilde{\chi}$ is the complex conjugate to character χ. Further progress is due to Hilbert and his *syzygy theorem* [4]. For our purpose it is important that $M^G(t)$ is a rational polynomial

$$M^G(t) = \frac{N^G(t)}{\prod_{l=1}^{n}(1 - t^{d_l})}, \quad N^G(t) = \sum_{k=0} Q(k, G)\, t^k. \tag{50}$$

The formula (50) is very convenient for expressing the function $P(s, G)$ in terms of the Sylvester waves $W(s, \mathbf{d}^m)$. Recalling the definition (1) of the generating function $F(t, \mathbf{d}^m)$ consider a general term $t^k F(t, \mathbf{d}^m)$ of the Molien function (50)

$$t^k F(t, \mathbf{d}^m) = \sum_{s=0}^{\infty} W(s, \mathbf{d}^m)t^{s+k} = \sum_{s=k}^{\infty} W(s-k, \mathbf{d}^m)t^s, \tag{51}$$

so that the corresponding partition function is $W(s-k, \mathbf{d}^m)$, which implies that the number $P(s, G)$ of all homogeneous invariants of degree s for the finite group G can be expressed through the simple relation

$$P(s, G) = \sum_{k=0}^{s} Q(k, G)W(s-k, \mathbf{d}^m)\,. \tag{52}$$

We consider several instructive examples for which the explicit expression of the Molien function $M^G(t)$ and the corresponding number of homogeneous invariants $P(s, G)$ are given.

1. Alternating group A_n generated by its natural n-dimensional representation, $|\mathsf{A}_n| = n!/2$.

$$M_{\mathsf{A}_n}(t) = \left[1 + t^{\binom{n}{2}}\right]\prod_{k=1}^{n}\frac{1}{1-t^k}\,.$$

$$P(s, \mathsf{A}_n) = W(s, \{\overline{n}\}) + W\left(s - \frac{n(n-1)}{2}, \{\overline{n}\}\right). \tag{53}$$

The group A_n acts on the Euclidean vector space $\mathbb{R}^n$.

2. Group G_2 generated by matrix $\begin{pmatrix} \rho_n & 0 \\ 0 & \rho_n^{-1} \end{pmatrix}$, where $\rho_n = e^{2\pi i/n}$ is a primitive n-th root of unity, $|\mathsf{G}_2| = n$.

$$M_{\mathsf{G}_2}(t) = \frac{1+t^n}{(1-t^2)(1-t^n)} \, .$$
$$P(s, \mathsf{G}_2) = W(s, \{2, n\}) + W(s-n, \{2, n\}). \tag{54}$$

G_2 is isomorphic as an abstract group to the cyclic group Z_n acting on the Euclidean vector space $\mathbb{R}^2$.

3. Group G_3 generated by the matrices $\begin{pmatrix} \rho_n & 0 \\ 0 & \rho_n^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $|\mathsf{G}_3| = 2n$.

$$M_{\mathsf{G}_3}(t) = \frac{1}{(1-t^2)(1-t^n)}, \quad P(s, \mathsf{G}_3) = W(s, \{2, n\}) \tag{55}$$

G_3 is isomorphic as an abstract group to the dihedral group I_n acting on Euclidean vector space $\mathbb{R}^2$.

4. Group G_4 generated by $(n \times n)$-diagonal matrix $\mathrm{diag}(-1, -1, \ldots, -1)$, $|\mathsf{G}_4| = 2$.

$$M_{\mathsf{G}_4}(t) = \frac{1}{(1-t^2)^n} \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} t^{2k} \, .$$

$$P(s, \mathsf{G}_4) = \begin{cases} \displaystyle\sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k} W(s-2k, \mathbf{2}^n) = W(s, \mathbf{1}^n), & s = 0 \pmod 2, \\ 0, & s \neq 0 \pmod 2. \end{cases} \tag{56}$$

G_4 is isomorphic as an abstract group to the cyclic group Z_2 acting on the Euclidean vector space $\mathbb{R}^n$.

It is easy to see that both groups G_2 and G_4 acting on $\mathbb{R}^2$ give rise to the same Molien function and corresponding number of invariants

$$M_{\mathsf{Z}_2}(t) = \frac{1+t^2}{(1-t^2)^2}, \quad P(s, \mathsf{Z}_2) = \begin{cases} W(s, \mathbf{1}^2), & s = 0 \pmod 2, \\ 0, & s \neq 0 \pmod 2. \end{cases} \tag{57}$$

5. Group Q_{4n} generated by the matrices $\begin{pmatrix} \rho_{2n} & 0 \\ 0 & \rho_{2n}^{-1} \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, $|\mathsf{Q}_{4n}| = 4n$.

$$M_{\mathsf{Q}_{4n}}(t) = \frac{1+t^{2n+2}}{(1-t^4)(1-t^{2n})}, \tag{58}$$
$$P(s, \mathsf{Q}_{4n}) = \begin{cases} W\left(\frac{s}{2}, \{2, n\}\right) + W\left(\frac{s}{2} - n - 1, \{2, n\}\right), & s = 0 \pmod 2, \\ 0, & s \neq 0 \pmod 2. \end{cases}$$

In the case of quaternion group Q_8 formula (58) reduces to

$$M_{\mathsf{Q}_8}(t) = \frac{1+t^6}{(1-t^4)^2}, \quad P(s, \mathsf{Q}_8) = \begin{cases} W(s, \mathbf{1}^2)/2, & s = 0 \pmod 4, \\ 0, & s \neq 0 \pmod 4. \end{cases} \tag{59}$$

More sophisticated examples of finite groups one can find in Appendices A, B of the book [4].

8. Conclusion

1. The explicit expression for the restricted partition function $W(s, \mathbf{d}^m)$ and its quasiperiodic components $W_j(s, \mathbf{d}^m)$ (*Sylvester waves*) for a set of positive integers $\mathbf{d}^m = \{d_1, d_2, \ldots, d_m\}$ is derived. The formulas are represented as a finite sum over Bernoulli and Eulerian polynomials of higher order with periodic coefficients.
2. Each Sylvester wave $W_j(s, \mathbf{d}^m)$ satisfies the same recursive relation as the whole partition function $W(s, \mathbf{d}^m)$.
3. The application of restricted partition function to the problem of counting all algebraically independent invariants of the degree s which arise due to action of finite group G on the vector space V over the field of complex numbers is discussed.

Appendices

A. Symbolic notation

The symbolic technique for manipulating sums with binomial coefficients by expanding polynomials and then replacing powers by subscripts was developed in nineteenth century by Blissard. It has been known as symbolic notation and the classical umbral calculus [12]. This notation can be used [10] to prove interesting formulas not easily proved by other methods. An example of this notation is also found in [2] in section devoted to the Bernoulli polynomials $B_k(x)$.

The well-known formulas

$$B_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} B_k(x) y^{n-k}, \quad B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k},$$

are written symbolically as

$$B_n(x+y) = (B(x)+y)^n, \quad B_n(x) = (B+x)^n.$$

After the expansion the exponents of $B(x)$ and B are converted into the orders of the Bernoulli polynomial and the Bernoulli number, respectively:

$$[B(x)]^k \Rightarrow B_k(x), \quad B^k \Rightarrow B_k. \tag{A1}$$

We use this notation in its extended version suggested in [7] in order to make derivation more clear and intelligible. Nörlund introduced the Bernoulli polynomials of higher order defined through the recursion

$$B_n^{(m)}(x|\mathbf{d}^m) = \sum_{k=0}^{n} \binom{n}{k} d^k B_k(0) B_{n-k}^{(m-1)}(x|\mathbf{d}^{m-1}), \tag{A2}$$

starting from $B_n^{(1)}(x|d_1) = d_1^n B_n\left(\frac{x}{d_1}\right)$. In symbolic notation it takes form

$$B_n^{(m)}(x) = \left(d_m B(0) + B^{(m-1)}(x)\right)^n ,$$

and recursively reduces to the more symmetric form

$$B_n^{(m)}(x|\mathbf{d}^m) = \left(x + d_1\ {}^1B(0) + d_2\ {}^2B(0) + \cdots + d_m\ {}^mB(0)\right)^n = \left(x + \sum_{i=1}^{m} d_i\ {}^iB(0)\right)^n , \tag{A3}$$

where each $[{}^iB(0)]^k$ is converted into $B_k(0)$.

B. Partition function for two primes

The polynomial part is computed according to (8)

$$W_1(ap_1p_2, \{p_1, p_2\}) = \frac{1}{p_1p_2} B_1^{(2)}(ap_1p_2 + p_1 + p_2|\{p_1, p_2\}) = a + \frac{1}{2}\left(\frac{1}{p_1} + \frac{1}{p_2}\right). \tag{B1}$$

Two other waves read

$$W_{p_1}(ap_1p_2, \{p_1, p_2\}) = \frac{1}{p_1}\sum_{r=1}^{p_1-1} \frac{1}{1-\rho_{p_1}^r}, \quad W_{p_2}(ap_1p_2, \{p_1, p_2\}) = \frac{1}{p_2}\sum_{r=1}^{p_2-1} \frac{1}{1-\rho_{p_2}^r} . \tag{B2}$$

where we use the trivial identity $\rho_{p_1}^{ap_1p_2} = \rho_{p_2}^{ap_1p_2} = 1$. For computation of the sums in (B2) we start with the identity (see [17])

$$\prod_{r=0}^{m-1} \left(x - \rho_m^r\right) = x^m - 1. \tag{B3}$$

and differentiate it with respect to x, and divide by $x^m - 1$

$$\sum_{r=0}^{m-1} \frac{1}{x - \rho_m^r} = \frac{mx^{m-1}}{x^m - 1}. \tag{B4}$$

Subtracting $1/(x-1)$ from both sides of (B4) and taking a limit at $x \to 1$ we obtain

$$\sum_{r=1}^{m-1} \frac{1}{1 - \rho_m^r} = \frac{m-1}{2}. \tag{B5}$$

Using this result we have for the periodic waves in (B2)

$$W_{p_1}(ap_1p_2, \{p_1, p_2\}) = \frac{p_1 - 1}{2p_1}, \quad W_{p_2}(ap_1p_2, \{p_1, p_2\}) = \frac{p_2 - 1}{2p_2}\ . \tag{B6}$$

Acknowledgment We thank I. M. Gessel for information about Ref. [10]. The research was supported in part (LGF) by the Gileadi Fellowship program of the Ministry of Absorption of the State of Israel.

References

1. Andrews, G.E.: The theory of partitions. Encyclopedia of mathematics and its applications, V.2, Addison-Wesley, (1976)
2. Bateman, H., Erdelýi, A.: Higher transcendental functions, V.1, McGraw-Hill Book Co., NY (1953)
3. Beck, M., Gessel, I.M., Komatsu, T.: The polynomial part of a restriction partition function related to the frobenius problem. The Electronic Journal of Combinatorics **8**(N7) 1–5 (2001)
4. Benson, D.J.: Polynomial invariants of finite groups. Cambridge Univ. Press, Cambridge (1993)
5. Carlitz, L.: Eulerian numbers and polynomials of higher order. Duke Mathematical Journal **27**, 401–423 (1960)
6. Dickson, L.E.: History of the theory of numbers. Chelsea Pub. Co., NY **2**, 114 (1952)
7. Nörlund, N.E.: Mémoire sur les polynomes de Bernoulli. Acta Mathematica **43**, 121–196 (1922)
8. Fel, L.G., Rubinstein, B.Y.: Sylvester waves in the coxeter groups. Ramanujan Journal **6**, 307–329 (2002)
9. Frobenius, F.G.: Über die Bernoullischen Zahlen und die Eulerischen polynome, Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin 809–847, (1910)
10. Gessel, I.M.: Applications of the classical umbral calculus. To appear in Algebra Universalis
11. Rademacher, H.: On the partition function $p(n)$. Proc. London Math. Soc. **43** 241–254 (1937)
12. Roman, S., Rota, G.-C.: The umbral calculus. Adv. Math. **27**, 95–188 (1978)
13. Shepard G.C., Todd, J.A.: Finite unitary reflection groups. Canad. J. Math. **6**, 274–304 (1954)
14. Stanley, R. P.: Invariants of finite groups and their applications to combinatorics. Bulletin of Amer. Math. Soc. **1**, 475–511 (1979)
15. Sylvester, J.J.: On the partition of numbers. Quarterly J. Math. **1**, 141–152 (1857)
16. Sylvester, J.J.: On subinvariants, i.e. semi-invariants to binary quantics of an unlimited order. With an excursus on rational fractions and partitions. Amer. J. Math. **5**, 79–136 (1882)
17. Vandiver, H.S.: An arithmetical theory of the Bernoulli numbers. Trans. Amer. Math. Soc. **51**, 502–531 (1942)