Extension of the Bernoulli and Eulerian Polynomials of Higher Order and Vector Partition Function

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Abstract

Following the ideas of L. Carlitz we introduce a generalization of the Bernoulli and Eulerian polynomials of higher order to vectorial index and argument. These polynomials are used for computation of the *vector partition function* $W(\mathbf{s},\mathbf{D})$, *i.e.*, a number of integer solutions to a linear system $\mathbf{x} \geq 0, \quad \mathbf{D}\mathbf{x} = \mathbf{s}$. It is shown that $W(\mathbf{s},\mathbf{D})$ can be expressed through the *vector* Bernoulli polynomials of higher order.

1 Introduction

The history of the Bernoulli polynomials $B_k(x)$ counts more than 250 years, as L. Euler first studied them for arbitrary values of the argument. He introduced the Euler polynomials $E_k(x)$, and relation between these functions was established in the end of nineteenth century by L. Saalschütz. The generalization of the Euler polynomials to the so-called Eulerian polynomials $H_k(x,\rho)$ was made by Frobenius [7] several years later.

N. Nörlund [9] introduced the Bernoulli $B_k^{(m)}(x|\mathbf{d}^m)$ and Euler polynomials $E_k^{(m)}(x|\mathbf{d}^m)$ of higher order adding m parameters. Similar extension for the Eulerian polynomials $H_k^{(m)}(x,\boldsymbol{\rho}^m|\mathbf{d}^m)$ was made by L. Carlitz in [4].

The Bernoulli and Eulerian polynomials of higher order appear to be useful for the description of the *restricted partition function* $W(s,\mathbf{d}^m)$, which is a number of integer nonnegative solutions of Diophantine equation $\mathbf{d}^m \cdot \mathbf{x} = s$ (see [11]). The author showed in [10] that $W(s,\mathbf{d}^m)$ may be written as a finite sum of the Bernoulli polynomial of higher order multiplied by *prime circulator* introduced by A. Cayley (see [6]). The short derivation of this result is given in Section 3.

Carlitz suggested in [5] another extension of the Bernoulli $B_{\mathbf{k}}(\mathbf{x})$ and Eulerian polynomials $H_{\mathbf{k}}(\mathbf{x},\rho)$ to vectorial arguments and indices.

In this work we combine both abovementioned approaches to introduce a new class of polynomials, which we call *vector* Bernoulli $B_{\mathbf{k}}^{(l,m)}(\mathbf{x}|\mathbf{D})$ and Eulerian $H_{\mathbf{k}}^{(l,m)}(\mathbf{x},\boldsymbol{\rho}|\mathbf{D})$ polynomials of higher order. We define the generating functions for these polynomials, find the recursive and symmetry relations, and find a new relation between these polynomials. Using the vector polynomials of higher order in Section 4 we find an explicit formula for the *vector partition function* $W(\mathbf{s},\mathbf{D})$, defined as a number of integer nonnegative solutions to the linear system $\mathbf{x} \geq 0, \quad \mathbf{D}\mathbf{x} = \mathbf{s}$, where $\mathbf{D}$ denotes a nonnegative non-degenerate integer matrix. It appears that similar to the scalar case the vector partition function may be written as a finite sum of the vector Bernoulli polynomials of higher order multiplied by prime circulators of vector index and argument. The solution gives the vector partition function in every chamber of the system, as well as it determines the shape of each individual chamber.

2 Bernoulli and Eulerian Polynomials and Their Generalizations

In the Section 2.1 the definition and main properties of the (regular) Bernoulli $B_k(x)$ and Eulerian $H_k(x,\rho)$ polynomials are presented. Their generalization to the polynomials of higher orders $B_k^{(m)}(x|\mathbf{d}^m)$ and $H_k^{(m)}(x,\boldsymbol{\rho}^m|\mathbf{d}^m)$, recursive and symmetry properties and some useful formulas are presented in Section 2.2. Extension of the regular polynomials to vectorial arguments and indices $B_{\mathbf{k}}(\mathbf{x})$ and $H_{\mathbf{k}}(\mathbf{x},\rho)$ described in the Section 2.3. Finally, in the Section 2.4 combining both extensions we introduce a new class of the polynomials $B_{\mathbf{k}}^{(l,m)}(\mathbf{x}|\mathbf{D})$ and $H_{\mathbf{k}}^{(l,m)}(\mathbf{x},\boldsymbol{\rho}|\mathbf{D})$ and consider their properties.

2.1 Bernoulli and Eulerian Polynomials

Define the Bernoulli polynomial $B_k(x)$ through the generating function

$$e^{xt}\frac{t}{e^t-1}=\sum_{k=0}^{\infty}B_k(x)\frac{t^k}{k!}.$$

The Bernoulli numbers are defined as $B_k=B_k(0)$. It is known that the *multiplication formula* holds for these polynomials:

$$\sum_{r=0}^{p-1}B_k\left(x+\frac{r}{p}\right)=p^{1-k}B_k(px);$$

this relation can be used as an alternative definition [8] of the Bernoulli polynomials. Another remarkable property of the Bernoulli polynomials is that they can be presented as a symbolic power

$$B_k(x)=(B+x)^k=\sum_{i=0}^{k}\binom{k}{i}B_ix^{k-i},$$

where after the expansion the powers of B are transformed into the indices.

The Eulerian polynomials were considered by Frobenius in [7], their generating function reads

$$e^{xt}\frac{1-\rho}{e^t-\rho}=\sum_{k=0}^{\infty}H_k(x,\rho)\frac{t^k}{k!},\quad (\rho\neq 1).$$

For $\rho=-1$ the Eulerian polynomials reduce to the Euler polynomials $E_k(x)$. The former also admits the symbolic form

$$H_k(x,\rho)=(H(\rho)+x)^k,$$

where $H_k(\rho)\equiv H_k(0,\rho)$ are the Eulerian numbers. For $\rho^p=1$ we have an additional relation [4] between the Bernoulli and Eulerian polynomials

$$p^{k-1}\sum_{r=0}^{p-1}\rho^{-r}B_k\left(x+\frac{r}{p}\right)=\frac{k\rho}{1-\rho}H_{k-1}(px,\rho). \quad (1)$$

2.2 Bernoulli and Eulerian Polynomials of Higher Order

Nörlund [9] introduced the Bernoulli polynomials of higher order $B^{(m)}(x|\mathbf{d}^m)$ adding m parameters $\mathbf{d}^m = \{d_1, d_2, \ldots, d_m\}$ by means of the generating function:

$$e^{xt}\prod_{i=1}^{m}\frac{d_i t}{e^{d_i t}-1} = \sum_{k=0}^{\infty} B_k^{(m)}(x|\mathbf{d}^m)\frac{t^k}{k!}.$$

The regular Bernoulli polynomials are expressed through the relation

$$B_k^{(1)}(x|d) = d^k B_k(x).$$

Nörlund also found the multiplication formulas

$$\sum_{r=0}^{p-1} B_k^{(m)}\left(x+\frac{r}{p}\sigma(\mathbf{d}^m)|\mathbf{d}^m\right) = p^{m-k}B_k^{(m)}(px|\mathbf{d}^m), \quad \sigma(\mathbf{d}^m) = \sum_{i=1}^{m} d_i,$$

$$\sum_{r_i=0}^{p_i-1} B_k^{(m)}(x+\mathbf{r}\cdot\mathbf{d}^m|\{p_1d_1,\ldots,p_md_m\}) = \pi(\mathbf{p}^m)B_k^{(m)}(x|\mathbf{d}^m), \quad \pi(\mathbf{p}^m) = \prod_{i=1}^{m} p_i,$$

where $\mathbf{r}\cdot\mathbf{d}^m = \sum_{i=1}^m r_i d_i$ denotes the scalar product. Nörlund gave the symbolic expression

$$B_k^{(m)}(x|\mathbf{d}^m) = (x+\sum_{i=1}^{m} d_i\, {}^iB)^k,$$

where again after the expansion powers of ${}^iB$ are converted into the indices of the Bernoulli numbers. This relation implies a binomial formula:

$$\sum_{k=0}^{n}\binom{n}{k}B_k^{(m)}(x|\mathbf{d}^m)B_{n-k}^{(l)}(y|\mathbf{d}^l) = B_n^{(m+l)}(x+y|\mathbf{d}^{m+l}). \tag{2}$$

The following symmetry relation is valid

$$B_k^{(m)}(x|\mathbf{d}^m) = (-1)^k B_k^{(m)}(-x-\sigma(\mathbf{d}^m)|\mathbf{d}^m), \tag{3}$$

and the recursive relation holds for the Bernoulli polynomials

$$B_k^{(m)}(x+d_m|\mathbf{d}^m) - B_k^{(m)}(x|\mathbf{d}^m) = kd_m B_{k-1}^{(m-1)}(x|\mathbf{d}^{m-1}). \tag{4}$$

The Eulerian polynomials of higher order $H^{(m)}(x,\boldsymbol{\rho}^m|\mathbf{d}^m)$ were considered by Carlitz in [4]. Their generating function reads

$$e^{xt}\prod_{i=1}^{m}\frac{1-\rho_i}{e^{d_i t}-\rho_i} = \sum_{k=0}^{\infty} H_k^{(m)}(x,\boldsymbol{\rho}^m|\mathbf{d}^m)\frac{t^k}{k!},$$

and they admit the following symbolic notation:

$$H_k^{(m)}(x,\boldsymbol{\rho}^m|\mathbf{d}^m) = (x+\sum_{i=1}^{m} d_i H(\rho_i))^k.$$

When $\rho_j^{p_j} = 1,\ \rho_j \neq 1,\ j = 1, 2, \ldots, m$ we have the analog of (1)

$$\frac{1}{\pi(\mathbf{p}^m)} \sum_{r_j=0}^{p_j-1} \rho_j^{-r_j} B_k^{(m)}(x + \mathbf{r} \cdot \mathbf{d}^m | \{p_1 d_1, p_2 d_2, \ldots, p_m d_m\})$$
$$= \frac{k!}{(k-m)!} \left(\prod_{i=1}^{m} \frac{\rho_i d_i}{1-\rho_i} \right) H_{k-m}^{(m)}(x, \boldsymbol{\rho}^m | \mathbf{d}^m). \qquad (5)$$

2.3 Vector Bernoulli and Eulerian Polynomials

Carlitz suggested another extension of the Bernoulli and Eulerian polynomials. He introduced [5] the Bernoulli polynomials $B_{\mathbf{k}}(\mathbf{x})$ of the vector argument $\mathbf{x} = \{x_1, \ldots, x_l\}$ and vector index $\mathbf{k} = \{k_1, \ldots, k_l\}$ through the generating function

$$e^{\mathbf{x}\cdot\mathbf{t}} \frac{\sum_{i=1}^{l} t_i}{e^{\sum_{i=1}^{l} t_i} - 1} = \sum_{\mathbf{k}} B_{\mathbf{k}}(\mathbf{x}) \frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!},$$

where the summation is over all $k_1, \ldots, k_l$ and $\mathbf{x} \cdot \mathbf{t} = \sum_{i=1}^{l} x_i t_i$ is a scalar product. Here and below we use the notation

$$\mathbf{t}^{\mathbf{k}} = \prod_{i=1}^{l} t_i^{k_i}, \quad \mathbf{k}! = \prod_{i=1}^{l} k_i!, \quad \binom{\mathbf{n}}{\mathbf{k}} = \prod_{i=1}^{l} \binom{n_i}{k_i}.$$

The multiplication theorem is established in the form

$$\sum_{r=0}^{p-1} B_{\mathbf{k}} \left(\mathbf{x} + \frac{r}{p} \right) = p^{1-|\mathbf{k}|} B_{\mathbf{k}}(p\mathbf{x}), \quad |\mathbf{k}| = \sum_{i=1}^{l} k_i.$$

The Eulerian polynomials $H_{\mathbf{k}}(\mathbf{x}, \rho)$ of the vector argument and index are defined as

$$e^{\mathbf{x}\cdot\mathbf{t}} \frac{1-\rho}{e^{\sum_{i=1}^{l} t_i} - \rho} = \sum_{\mathbf{k}=0}^{\infty} H_{\mathbf{k}}(\mathbf{x}, \rho) \frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!},$$

and the analog of (1) for $\rho^p = 1$ reads:

$$p^{|\mathbf{k}|-1} \sum_{r=0}^{p-1} \rho^{-r} B_{\mathbf{k}} \left(\mathbf{x} + \frac{r}{p} \right) = \frac{\rho}{1-\rho} \sum_{i=1}^{l} k_i H_{\mathbf{k}-\mathbf{1}^{(i)}}(p\mathbf{x}, \rho),$$

where $\mathbf{1}^{(i)}$ is vector with the components $\mathbf{1}_m^{(i)} = \delta_{im}$, and δ_{im} is the Kroneker delta.

Here and below we will use the notion of the *vector Bernoulli* $B_{\mathbf{k}}(\mathbf{x})$ *and Eulerian* $H_{\mathbf{k}}(\mathbf{x}, \rho)$ *polynomials* remembering that they are scalar functions of their arguments.

2.4 Vector Bernoulli and Eulerian Polynomials of higher order

In this work we make one more step in generalization of the Bernoulli and Eulerian polynomials which naturally arises from the approaches discussed above. We consider *vector Bernoulli* $B_{\mathbf{k}}^{(l,m)}(\mathbf{x}|\mathbf{D}^m)$ *and Eulerian* $H_{\mathbf{k}}^{(l,m)}(\mathbf{x}, \boldsymbol{\rho}^m|\mathbf{D}^m)$ *polynomials of higher order*, where $\mathbf{D}^m$ denotes $(l \times m)$ matrix.

The vector Bernoulli polynomials of higher order are defined by means of the generating function:

$$e^{\mathbf{x}\cdot\mathbf{t}}\prod_{j=1}^{m}\frac{\sum_{i=1}^{l}t_iD_{ij}}{e^{\sum_{i=1}^{l}t_iD_{ij}}-1}=\sum_{\mathbf{k}}B_{\mathbf{k}}^{(l,m)}(\mathbf{x}|\mathbf{D}^m)\frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}\ .$$

Denoting the columns of the matrix $\mathbf{D}^m$ as $\{\mathbf{c}_1,\ldots,\mathbf{c}_m\}$, we can rewrite the above definition as

$$e^{\mathbf{x}\cdot\mathbf{t}}\prod_{j=1}^{m}\frac{\mathbf{t}\cdot\mathbf{c}_j}{e^{\mathbf{t}\cdot\mathbf{c}_j}-1}=\sum_{\mathbf{k}}B_{\mathbf{k}}^{(l,m)}(\mathbf{x}|\mathbf{D}^m)\frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}\ . \tag{6}$$

The analog of the recursive relation (4) reads

$$B_{\mathbf{k}}^{(l,m)}(\mathbf{x}+\mathbf{c}_m|\mathbf{D}^m)-B_{\mathbf{k}}^{(l,m)}(\mathbf{x}|\mathbf{D}^m)=\sum_{i=1}^{l}k_iD_{im}B_{\mathbf{k}-\mathbf{1}^{(i)}}^{(l,m-1)}(\mathbf{x}|\mathbf{D}^{m-1})\ .$$

One can check the validity of an analog of symmetry relation (3)

$$B_{\mathbf{k}}^{(l,m)}(\mathbf{x}|\mathbf{D}^m)=(-1)^{|\mathbf{k}|}B_{\mathbf{k}}^{(l,m)}(-\mathbf{x}-\boldsymbol{\sigma}(\mathbf{D}^m)|\mathbf{D}^m),\quad \boldsymbol{\sigma}(\mathbf{D}^m)=\sum_{j=1}^{m}\mathbf{c}_j.$$

Using (6) we obtain the vector generalization of the binomial formula (2)

$$\sum_{\mathbf{k}}^{\mathbf{n}}\binom{\mathbf{n}}{\mathbf{k}}B_{\mathbf{k}}^{(l,m_1)}(\mathbf{x}|\mathbf{D}^{m_1})B_{\mathbf{n}-\mathbf{k}}^{(l,m_2)}(\mathbf{y}|\mathbf{D}^{m_2})=B_{\mathbf{n}}^{(l,m_1+m_2)}(\mathbf{x}+\mathbf{y}|\mathbf{D}^{m_1+m_2}). \tag{7}$$

The vector Eulerian polynomials of higher order have the generating function of the form:

$$e^{\mathbf{x}\cdot\mathbf{t}}\prod_{j=1}^{m}\frac{1-\rho_j}{e^{\mathbf{t}\cdot\mathbf{c}_j}-\rho_j}=\sum_{\mathbf{k}}H_{\mathbf{k}}^{(l,m)}(\mathbf{x},\boldsymbol{\rho}^m|\mathbf{D}^m)\frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}\ ,$$

where $\boldsymbol{\rho}^m=\{\rho_1,\ldots,\rho_m\}$. Consider a relation between the vector Bernoulli and Eulerian polynomials of higher order similar to (5). For p_j such that $\rho_j^{p_j}=1$ we have the following expression:

$$\begin{aligned}
&\sum_{\mathbf{k}}\sum_{r_j=0}^{p_j-1}\rho_j^{-r_j}B_{\mathbf{k}}^{(l,m)}(\mathbf{x}+\sum_{j=1}^{m}r_j\mathbf{c}_j|\{p_1\mathbf{c}_1,\ldots,p_m\mathbf{c}_m\})\frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}\\
=\ &\prod_{j=1}^{m}\sum_{r_j=0}^{p_j-1}\rho_j^{-r_j}\exp[(\mathbf{x}+\sum_{j=1}^{m}r_j\mathbf{c}_j)\cdot\mathbf{t}]\left(\frac{p_j\mathbf{c}_j\cdot\mathbf{t}}{e^{p_j\mathbf{c}_j\cdot\mathbf{t}}-1}\right)\\
=\ &\left(\prod_{j=1}^{m}\frac{p_j\rho_j}{1-\rho_j}(\mathbf{c}_j\cdot\mathbf{t})\right)e^{\mathbf{x}\cdot\mathbf{t}}\prod_{j=1}^{m}\frac{1-\rho_j}{e^{\mathbf{c}_j\cdot\mathbf{t}}-\rho_j}\\
=\ &\pi(\mathbf{p}^m)\left(\prod_{j=1}^{m}\frac{\rho_j}{1-\rho_j}(\mathbf{c}_j\cdot\mathbf{t})\right)\sum_{\mathbf{k}}H_{\mathbf{k}}^{(l,m)}(\mathbf{x},\boldsymbol{\rho}^m|\mathbf{D}^m)\frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}.
\end{aligned} \tag{8}$$

3 Restricted Partition Function

The restricted partition function $W(s, \mathbf{d}^m) \equiv W(s, \{d_1, d_2, \ldots, d_m\})$ is a number of partitions of an integer s into positive integers $\{d_1, d_2, \ldots, d_m\}$, each not greater than s. The generating function for $W(s, \mathbf{d}^m)$ has a form

$$\prod_{i=1}^{m} \frac{1}{1-t^{d_i}} = \sum_{s=0}^{\infty} W(s, \mathbf{d}^m)\, t^s \ ,$$

where $W(s, \mathbf{d}^m)$ satisfies the basic recursive relation

$$W(s, \mathbf{d}^m) - W(s - d_m, \mathbf{d}^m) = W(s, \mathbf{d}^{m-1}) \ . \tag{9}$$

Note a similarity of (9) to the recursive relation (4) for the Bernoulli polynomials of higher order. Sylvester found a symmetry property of the partition function:

$$W(s, \mathbf{d}^m) = (-1)^{m-1} W(-s - \sigma(\mathbf{d}^m), \mathbf{d}^m), \quad \sigma(\mathbf{d}^m) = \sum_{i=1}^{m} d_i. \tag{10}$$

He proved [12] a statement about splitting of the partition function into periodic and non-periodic parts and showed that the restricted partition function may be presented as a sum of "waves", which we call the *Sylvester waves*

$$W(s, \mathbf{d}^m) = \sum_{j=1} W_j(s, \mathbf{d}^m) \ , \tag{11}$$

where summation runs over all distinct factors of the elements of the set $\mathbf{d}^m$. The wave $W_j(s, \mathbf{d}^m)$ is a quasipolynomial in s closely related to prime roots ρ_j of unity. The wave $W_j(s, \mathbf{d}^m)$ is a coefficient of t^{-1} in the series expansion in ascending powers of t of the generator

$$F_j^m(s,t) = \sum_{\rho_j} \frac{\rho_j^{-s} e^{st}}{\prod_{k=1}^{m}\left(1 - \rho_j^{d_k} e^{-d_k t}\right)} \ . \tag{12}$$

The summation is made over all prime roots of unity $\rho_j = \exp(2\pi i n/j)$ for n relatively prime to j (including unity) and smaller than j. It is easy to check by straightforward calculation that the recursive relation

$$F_j^m(s,t) - F_j^m(s - d_m, t) = F_j^{m-1}(s,t) \tag{13}$$

holds for any generator $F_j^m(s,t)$, impliyng the validity of (9) for each Sylverster wave $W_j(s, \mathbf{d}^m)$. The generator satisfies the following symmetry property

$$F_j^m(s,t) = (-1)^m F_j^m(-s - \sigma(\mathbf{d}^m), -t),$$

which implies the validity of (10) for the residue of $F_j^m(s,t)$.

In [11] the explicit expression for the Sylvester wave of the arbitrary period is given through the Bernoulli and Eulerian polynomials of higher order. Using (5) it was shown in [10] that it is possible to express the Sylvester wave as a finite sum of the Bernoulli polynomials of higher order only. Here we present a short derivation of this result.

Assuming that the vector $\mathbf{d}^m$ has ω_j components divisible by j, sort the elements of $\mathbf{d}^m$ in such way that the j-divisible integers come first. The generator (12) can be written as a product

$$F_j(s,t) = \sum_{\rho_j} \frac{e^{st}}{\prod_{i=1}^{\omega_j}(1 - e^{-d_i t})} \times \frac{\rho_j^{-s}}{\prod_{i=\omega_j+1}^{m}(1 - \rho_j^{d_i} e^{-d_i t})} \ .$$

Consider a modified generator $\tilde{F}_j^m(s,t) = \pi(\mathbf{d}^m) t^m F_j^m(s,t)$, for which using the notation $\rho^{\mathbf{d}^m} = \{\rho^{d_1}, \rho^{d_2}, \ldots, \rho^{d_m}\}$ we have

$$\begin{aligned}
\tilde{F}_j^m(s,t) &= t^{m-\omega_j} \sum_{\rho_j} \rho_j^{-s} \prod_{i=1}^{\omega_j} \frac{d_i t}{e^{d_i t}-1} \cdot e^{(s+\sigma(\mathbf{d}^m))t} \prod_{i=\omega_j+1}^{m} \frac{d_i}{e^{d_i t} - \rho_j^{d_i}} \\
&= t^{m-\omega_j} \sum_{\rho_j} \rho_j^{-s-\sigma(\mathbf{d}^{m-\omega_j})} \prod_{i=1}^{\omega_j} \frac{d_i t}{e^{d_i t}-1} \cdot e^{(s+\sigma(\mathbf{d}^m))t} \prod_{i=\omega_j+1}^{m} \frac{1-\rho_j^{d_i}}{e^{d_i t} - \rho_j^{d_i}} \cdot \frac{d_i \rho_j^{d_i}}{1-\rho_j^{d_i}} \\
&= t^{m-\omega_j} \sum_{\rho_j} \rho_j^{-s-\sigma(\mathbf{d}^{m-\omega_j})} \cdot \sum_{n_1=0}^{\infty} B_{n_1}^{(\omega_j)}(0|\mathbf{d}^{\omega_j}) \frac{t^{n_1}}{n_1!} \qquad (14) \\
&\times \prod_{i=\omega_j+1}^{m} \frac{d_i \rho_j^{d_i}}{1-\rho_j^{d_i}} \cdot \sum_{n_2=0}^{\infty} H_{n_2}^{(m-\omega_j)}(s+\sigma(\mathbf{d}^m), \rho_j^{\mathbf{d}^{m-\omega_j}} | \mathbf{d}^{m-\omega_j}) \frac{t^{n_2}}{n_2!}.
\end{aligned}$$

Now we employ (5), noting that for $\omega+1 \le i \le m$ all $p_i = j$, to obtain

$$\begin{aligned}
&\prod_{i=\omega_j+1}^{m} \frac{d_i \rho_j^{d_i}}{1-\rho_j^{d_i}} \cdot \sum_{n_2=0}^{\infty} H_{n_2}^{(m-\omega_j)}(s+\sigma(\mathbf{d}^m), \rho_j^{\mathbf{d}^{m-\omega_j}} | \mathbf{d}^{m-\omega_j}) \frac{t^{n_2}}{n_2!} \\
= &\sum_{n_2=0}^{\infty} \frac{t^{n_2}}{(m-\omega_j+n_2)! j^{m-\omega_j}} \sum_{r_i=0}^{j-1} \rho_j^{-r_i d_i} B_{m-\omega_j+n_2}^{(m-\omega_j)}(s+\sigma(\mathbf{d}^m) + \mathbf{r}\cdot\mathbf{d}^{m-\omega_j} | j\mathbf{d}^{m-\omega_j}).
\end{aligned}$$

Inserting the above expression into (14) and using (2) we obtain

$$\tilde{F}_j^m(s,t) = j^{-(m-\omega_j)} \sum_{\rho_j} \sum_{r_i=0}^{j-1} \rho_j^{-s-\mathbf{r}\cdot\mathbf{d}^{m-\omega_j}-\sigma(\mathbf{d}^m)} \sum_n B_n^{(m)}(s+\sigma(\mathbf{d}^m) + \mathbf{r}\cdot\mathbf{d}^{m-\omega_j} | \mathbf{d}_j^m) \frac{t^n}{n!}, \qquad (15)$$

where we use a shorthand notation $\mathbf{d}_j^m$ for *j-modified* set of summands defined as union of subset $\mathbf{d}^{\omega_j}$ of summands divisible by j and the remaining part $\mathbf{d}^{m-\omega_j}$ multiplied by j

$$\mathbf{d}_j^m = \mathbf{d}^{\omega_j} \cup j\mathbf{d}^{m-\omega_j} = \{d_1, \ldots, d_{\omega_j}, jd_{\omega_j+1}, \ldots, jd_m\},$$

so that $\mathbf{d}_j^m$ is divisible by j. From (15) the expression for $F_j^m(s,t)$ follows:

$$F_j^m(s,t) = \frac{1}{\pi(\mathbf{d}^m) j^{m-\omega_j}} \sum_{\rho_j} \sum_{r_i=0}^{j-1} \rho_j^{-s-\mathbf{r}\cdot\mathbf{d}^{m-\omega_j}-\sigma(\mathbf{d}^m)} \sum_n B_n^{(m)}(s+\sigma(\mathbf{d}^m) + \mathbf{r}\cdot\mathbf{d}^{m-\omega_j} | \mathbf{d}_j^m) \frac{t^{n-m}}{n!}. \quad (16)$$

Setting in (16) $n = m-1$ we arrive at the expression for the Sylvester wave:

$$W_j(s,\mathbf{d}^m) = \frac{1}{(m-1)!\, \pi(\mathbf{d}^m)\, j^{m-\omega_j}} \sum_{r_i=0}^{j-1} B_{m-1}^{(m)}(s+\sigma(\mathbf{d}^m) + \mathbf{r}\cdot\mathbf{d}^{m-\omega_j} | \mathbf{d}_j^m) \sum_{\rho_j} \rho_j^{-s-\mathbf{r}\cdot\mathbf{d}^{m-\omega_j}-\sigma(\mathbf{d}^m)}.$$

Introduce a notation

$$\Psi_j(s) = \sum_{\rho_j} \rho_j^s$$

for the *prime radical circulator* (see [6]). For prime j it is given by

$$\Psi_j(s) = \begin{cases} \phi(j), & s = 0 \pmod j, \\ \mu(j), & s \neq 0 \pmod j, \end{cases}$$

where $\phi(j)$ and $\mu(j)$ denote Euler totient and Möbius functions. Considering j as a product of powers of distinct prime factors

$$j = \prod_k p_k^{\alpha_k},$$

one may easily check that for integer values of s

$$\Psi_j(s) = \prod_k p_k^{\alpha_k-1}\Psi_{p_k}\left(\frac{s}{p_k^{\alpha_k-1}}\right),$$

where $\Psi_k(s) = 0$ for non-integer values of s.

Noting that for j-modified set $\mathbf{d}_j^m$ we have

$$\pi(\mathbf{d}_j^m) = j^{m-\omega_j}\pi(\mathbf{d}^m), \quad \sigma(\mathbf{d}_j^m) = \sigma(\mathbf{d}^m) + (j-1)\sum_{i=\omega_j+1}^{m} d_i,$$

and using the prime circulator notation we can write the Sylvester wave in a form

$$W_j(s,\mathbf{d}^m) = \frac{1}{(m-1)!\;\pi(\mathbf{d}_j^m)}\sum_{r_i=0}^{j-1} B_{m-1}^{(m)}(s+\sigma(\mathbf{d}_j^m) - \mathbf{r}\cdot\mathbf{d}^{m-\omega_j}|\mathbf{d}_j^m)\Psi_j(s-\mathbf{r}\cdot\mathbf{d}^{m-\omega_j}).$$

The polynomial part of the partition function corresponds to $j=1$ and equals to

$$W_1(s,\mathbf{d}^m) = \frac{1}{(m-1)!\;\pi(\mathbf{d}^m)} B_{m-1}^{(m)}(s+\sigma(\mathbf{d}^m)|\mathbf{d}^m). \tag{17}$$

The polynomial part for the j-modified set $\mathbf{d}_j^m$ for $j>1$ reads

$$W_1(s,\mathbf{d}_j^m) = \frac{1}{(m-1)!\;\pi(\mathbf{d}_j^m)} B_{m-1}^{(m)}(s+\sigma(\mathbf{d}_j^m)|\mathbf{d}_j^m). \tag{18}$$

Thus, the Sylvester wave for $j>1$ can be written as a linear superposition of the polynomial part of the j-modified set multiplied by the corresponding prime circulator:

$$W_j(s,\mathbf{d}^m) = \sum_{r_i=0}^{j-1} W_1(s-\mathbf{r}\cdot\mathbf{d}^{m-\omega_j},\mathbf{d}_j^m)\Psi_j(s-\mathbf{r}\cdot\mathbf{d}^{m-\omega_j}). \tag{19}$$

It is easy to see that for $j=1$ one has $\mathbf{d}_1^m \equiv \mathbf{d}^m$, $\Psi_1(s)=1$ and $\omega_1 = m$, so that summation signs disappear, and (19) reduces to (17).

4 Restricted Vector Partition Function

Consider a function $W(\mathbf{s},\mathbf{D}^m)$ counting the number of integer nonnegative solutions $\mathbf{x}\ge 0$ to the linear system $\mathbf{D}^m\cdot\mathbf{x} = \mathbf{s}$, where $\mathbf{D}^m$ is a nonnegative integer $(l\times m)$ matrix. The function $W(\mathbf{s},\mathbf{D}^m)$ is called *vector partition function* as it is natural generalization of the restricted partition function to the vector argument.

The generating function for the vector partition function reads

$$\prod_{i=1}^{m}\frac{1}{1-\mathbf{t}^{\mathbf{c}_i}} = \sum_{\mathbf{s}} W(\mathbf{s},\mathbf{D}^m)\mathbf{t}^{\mathbf{s}} = \sum_{\mathbf{s}} W(\mathbf{s},\{\mathbf{c}_1,\ldots,\mathbf{c}_m\})\mathbf{t}^{\mathbf{s}}\;, \tag{20}$$

where $\mathbf{c}_i$ denotes the i-th column of the matrix $\mathbf{D}^m$. It is easy to see that the vector analog of the recursive relation (9) holds

$$W(\mathbf{s},\mathbf{D}^m) - W(\mathbf{s}-\mathbf{c}_m,\mathbf{D}^m) = W(\mathbf{s},\mathbf{D}^{m-1}). \tag{21}$$

The symmetry property established in [1] is just a vector generalization of (10):

$$W(\mathbf{s},\mathbf{D}^m) = (-1)^{m-\mathrm{rank}\,\mathbf{D}^m} W(-\mathbf{s}-\boldsymbol{\sigma}(\mathbf{D}^m),\mathbf{D}^m), \quad \boldsymbol{\sigma}(\mathbf{D}^m) = \sum_{i=1}^m \mathbf{c}_i. \tag{22}$$

Similarly to the Sylvester splitting theorem (11) we write the vector partition function as a sum of *vector Sylvester waves*

$$W(\mathbf{s},\mathbf{D}^m) = \sum_{\mathbf{j}} W_{\mathbf{j}}(\mathbf{s},\mathbf{D}^m) = \sum_{j_1,\ldots,j_l} W_{\mathbf{j}}(\mathbf{s},\mathbf{D}^m), \tag{23}$$

where $\mathbf{j}$ denotes l-dimensional vector $(j_1,\ldots,j_l)$. The summation for each j_k runs over all distinct factors of the elements of the matrix $\mathbf{D}^m$. The vector analog of the generator (12) is written in the form (see [3])

$$F_{\mathbf{j}}^m(\mathbf{s},\mathbf{t}) = \sum_{\rho_{j_1},\rho_{j_2},\ldots,\rho_{j_l}} \frac{e^{\mathbf{s}\cdot\mathbf{t}}\boldsymbol{\rho}_{\mathbf{j}}^{-\mathbf{s}}}{\prod_{i=1}^m (1-\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i} e^{-\mathbf{c}_i\cdot\mathbf{t}})},$$

where

$$\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i} = \prod_{k=1}^l \rho_{j_k}^{D_{ki}}. \tag{24}$$

It satisfies the relation similar to (13)

$$F_{\mathbf{j}}^m(\mathbf{s},\mathbf{t}) - F_{\mathbf{j}}^m(\mathbf{s}-\mathbf{c}_m,\mathbf{t}) = F_{\mathbf{j}}^{m-1}(\mathbf{s},\mathbf{t})$$

for any generator $F_{\mathbf{j}}^m(\mathbf{s},\mathbf{t})$. The following relation aslo holds

$$F_{\mathbf{j}}^m(\mathbf{s},\mathbf{t}) = (-1)^m F_{\mathbf{j}}^m(-\mathbf{s}-\boldsymbol{\sigma}(\mathbf{D}^m),-\mathbf{t}).$$

The multidimensional residue of the generator gives $W_{\mathbf{j}}(\mathbf{s},\mathbf{D}^m)$ as the coefficient of $\mathbf{t}^{-\mathbf{1}} = \prod_{k=1}^l t_k^{-1}$, and each vector wave satisfies the relations (21) and (22).

Let an equality $\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i} = 1$ holds for $\omega_{\mathbf{j}}$ columns of the matrix $\mathbf{D}^m$, and sort the matrix in such way that these $\omega_{\mathbf{j}}$ columns come first, so that $1 \le i \le \omega_{\mathbf{j}}$. Introduce a homogeneous polynomial of degree m

$$P_m(\mathbf{t},\mathbf{D}^m) = \prod_{i=1}^m (\mathbf{c}_i\cdot\mathbf{t}) = \sum_{|\mathbf{N}|=m} C_{\mathbf{N}}(\mathbf{D}^m)\mathbf{t}^{\mathbf{N}},$$

and construct the modified generator

$$\tilde{F}_{\mathbf{j}}^m(\mathbf{s},\mathbf{t}) = P_m(\mathbf{t},\mathbf{D}^m) F_{\mathbf{j}}^m(\mathbf{s},\mathbf{t}),$$

which can be written as

$$\begin{aligned}
\tilde{F}_{\mathbf{j}}^m(\mathbf{s},\mathbf{t}) &= \sum_{\rho_{jn}} \boldsymbol{\rho}_{\mathbf{j}}^{-\mathbf{s}} \prod_{i=1}^{\omega_{\mathbf{j}}} \frac{\mathbf{c}_i\cdot\mathbf{t}}{e^{\mathbf{c}_i\cdot\mathbf{t}}-1} \times e^{(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}^m))\cdot\mathbf{t}} \prod_{i=\omega_{\mathbf{j}}+1}^m \frac{1-\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_j}}{e^{\mathbf{c}_i\cdot\mathbf{t}}-\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i}} \cdot \frac{\mathbf{c}_i\cdot\mathbf{t}}{1-\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i}} \\
&= \sum_{\rho_{jn}} \boldsymbol{\rho}_{\mathbf{j}}^{-\mathbf{s}-\boldsymbol{\sigma}(\mathbf{D}^{m-\omega_{\mathbf{j}}})} \sum_{\mathbf{n}_1} B_{\mathbf{n}_1}^{(l,\omega_{\mathbf{j}})}(0|\mathbf{D}^{\omega_{\mathbf{j}}}) \frac{\mathbf{t}^{\mathbf{n}_1}}{\mathbf{n}_1!} \\
&\times \prod_{i=\omega_{\mathbf{j}}+1}^m \frac{(\mathbf{c}_i\cdot\mathbf{t})\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i}}{1-\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i}} \sum_{\mathbf{n}_2} H_{\mathbf{n}_2}^{(l,m-\omega_{\mathbf{j}})}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}^m),\boldsymbol{\rho}_{\mathbf{j}}^{m-\omega_{\mathbf{j}}}|\mathbf{D}^{m-\omega_{\mathbf{j}}}) \frac{\mathbf{t}^{\mathbf{n}_2}}{\mathbf{n}_2!},
\end{aligned}$$

where elements of the vector $\boldsymbol{\rho}^{m-\omega_{\mathbf{j}}}$ are $\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i} \neq 1$ given by (24) for $\omega_{\mathbf{j}}+1 \leq i \leq m$. Using the relation (8) with p_i such that $\boldsymbol{\rho}_{\mathbf{j}}^{p_i\mathbf{c}_i} = 1$ we can write

$$\prod_{i=\omega_{\mathbf{j}}+1}^{m} \frac{(\mathbf{c}_i \cdot \mathbf{t})\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i}}{1-\boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i}} \sum_{\mathbf{n}_2} H_{\mathbf{n}_2}^{(l,m-\omega_{\mathbf{j}})}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}^m), \boldsymbol{\rho}_{\mathbf{j}}^{\mathbf{c}_i}|\mathbf{D}^{m-\omega_{\mathbf{j}}})\frac{\mathbf{t}^{\mathbf{n}_2}}{\mathbf{n}_2!}$$

$$= \frac{1}{\pi(\mathbf{p}^{m-\omega_{\mathbf{j}}})} \sum_{\mathbf{n}_2} \sum_{r_i=0}^{p_i-1} \boldsymbol{\rho}_{\mathbf{j}}^{-r_i\mathbf{c}_i} B_{\mathbf{n}_2}^{(l,m-\omega_{\mathbf{j}})}(\mathbf{s} + \sum_{i=\omega_{\mathbf{j}}+1}^{m} r_i\mathbf{c}_i|\{p_{\omega_{\mathbf{j}}+1}\mathbf{c}_{\omega_{\mathbf{j}}+1}, \ldots, p_m\mathbf{c}_m\})\frac{\mathbf{t}^{\mathbf{n}_2}}{\mathbf{n}_2!}.$$

Employing the binomial formula (7) we obtain $\tilde{F}_{\mathbf{j}}^m(\mathbf{s},\mathbf{t})$ in the form

$$\tilde{F}_{\mathbf{j}}^m(\mathbf{s},\mathbf{t}) = \frac{1}{\pi(\mathbf{p}^{m-\omega_{\mathbf{j}}})} \sum_{\mathbf{n}} \sum_{r_i=0}^{p_i-1} B_{\mathbf{n}}^{(l,m)}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}^m)+\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}|\mathbf{D}_{\mathbf{j}}^m)\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!} \cdot \sum_{\rho_{j_i}} \boldsymbol{\rho}_{\mathbf{j}}^{-\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}},$$

where $\mathbf{D}_{\mathbf{j}}^m$ is the $\mathbf{j}$-modified matrix of the form

$$\mathbf{D}_{\mathbf{j}}^m = \{\mathbf{c}_1, \ldots, \mathbf{c}_\omega, p_{\omega_{\mathbf{j}}+1}\mathbf{c}_{\omega_{\mathbf{j}}+1}, \ldots, p_m\mathbf{c}_m\}.$$

Noting that for the $\mathbf{j}$-modified matrix $\mathbf{D}_{\mathbf{j}}^m$

$$P_m(\mathbf{t},\mathbf{D}_{\mathbf{j}}^m) = \pi(\mathbf{p}^{m-\omega_{\mathbf{j}}})P_m(\mathbf{t},\mathbf{D}^m), \quad \boldsymbol{\sigma}(\mathbf{D}_{\mathbf{j}}^m) = \boldsymbol{\sigma}(\mathbf{D}^m) + \sum_{i=\omega_{\mathbf{j}}+1}^{m}(p_i-1)\mathbf{c}_i,$$

and using the vector prime circulator notation

$$\Psi_{\mathbf{c}}(\mathbf{s}) = \prod_{k=1}^{l} \Psi_{c_k}(s_k),$$

we write for the generator $F_{\mathbf{j}}^m(\mathbf{s},\mathbf{t})$

$$F_{\mathbf{j}}^m(\mathbf{s},\mathbf{t}) = P_m^{-1}(\mathbf{t},\mathbf{D}_{\mathbf{j}}^m) \sum_{\mathbf{n}} \sum_{r_i=0}^{p_i-1} B_{\mathbf{n}}^{(l,m)}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}_{\mathbf{j}}^m)-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}|\mathbf{D}_{\mathbf{j}}^m)\Psi_{\mathbf{j}}(\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}})\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}.$$

In the above expression only terms with $|\mathbf{n}| = m-l$ contribute to the vector Sylvester wave $W_{\mathbf{j}}(\mathbf{s},\mathbf{D}^m)$, which is found also as a constant term of $F_{\mathbf{j}}^m(\mathbf{s},\mathbf{t})\mathbf{t}^{\mathbf{1}}$, equal to a fraction with both numerator and denominator being homogeneous polynomials of degree m

$$W_{\mathbf{j}}(\mathbf{s},\mathbf{D}^m) = \lim_{\mathbf{t}\to\mathbf{0}} P_m^{-1}(\mathbf{t},\mathbf{D}_{\mathbf{j}}^m) \sum_{|\mathbf{n}|=m-l} \sum_{r_i=0}^{p_i-1} B_{\mathbf{n}}^{(l,m)}(\mathbf{s}'|\mathbf{D}_{\mathbf{j}}^m)\Psi_{\mathbf{j}}(\mathbf{s}')\frac{\mathbf{t}^{\mathbf{n}+\mathbf{1}}}{\mathbf{n}!}, \quad \mathbf{s}' = \mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}_{\mathbf{j}}^m)-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}.$$

It is convenient to write each vector Sylvester wave $W_{\mathbf{j}}(\mathbf{s},\mathbf{D}^m)$ as a sum of quasipolynomials, which we call *partial vector Sylvester waves*:

$$W_{\mathbf{j}}(\mathbf{s},\mathbf{D}^m) = \sum_{|\mathbf{n}|=m-l} W_{\mathbf{j}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}^m). \tag{25}$$

The number of different partial waves is found as a number of representations of an integer $m-l$ as a sum of l nonnegative integers, i.e., it is equal to

$$W(m-l,\mathbf{1}^l) = \binom{m-1}{m-l}.$$

The partial wave $W_{\mathbf{j}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}^m)$ is found as ($|\mathbf{n}|=m-l$)

$$W_{\mathbf{j}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}^m)=\lim_{\mathbf{t}\to\mathbf{0}} P_m^{-1}(\mathbf{t},\mathbf{D}_{\mathbf{j}}^m)\sum_{r_i=0}^{p_i-1} B_{\mathbf{n}}^{(l,m)}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}_{\mathbf{j}}^m)-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}|\mathbf{D}_{\mathbf{j}}^m)\Psi_{\mathbf{j}}(\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}})\frac{\mathbf{t}^{\mathbf{n}+\mathbf{1}}}{\mathbf{n}!}.$$

Using a parametrization $\mathbf{t}=\boldsymbol{\alpha}t$ we compute the above limit as

$$\begin{aligned}
W_{\mathbf{j}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}^m) &= P_m^{-1}(\boldsymbol{\alpha},\mathbf{D}_{\mathbf{j}}^m)\sum_{r_i=0}^{p_i-1} B_{\mathbf{n}}^{(l,m)}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}_{\mathbf{j}}^m)-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}|\mathbf{D}_{\mathbf{j}}^m)\Psi_{\mathbf{j}}(\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}})\frac{\boldsymbol{\alpha}^{\mathbf{n}+\mathbf{1}}}{\mathbf{n}!}\\
&= C_{\mathbf{n}}(\boldsymbol{\alpha},\mathbf{D}_{\mathbf{j}}^m)\sum_{r_i=0}^{p_i-1} B_{\mathbf{n}}^{(l,m)}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}_{\mathbf{j}}^m)-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}|\mathbf{D}_{\mathbf{j}}^m)\Psi_{\mathbf{j}}(\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}),
\end{aligned}$$

where

$$C_{\mathbf{n}}(\boldsymbol{\alpha},\mathbf{D}_{\mathbf{j}}^m)=\frac{\boldsymbol{\alpha}^{\mathbf{n}+\mathbf{1}}}{\mathbf{n}!P_m(\boldsymbol{\alpha},\mathbf{D}_{\mathbf{j}}^m)},\quad |\mathbf{n}|=m-l.$$

We define the polynomial part of the vector partition function for $\mathbf{j}=\mathbf{1}$, the corresponding partial polynomial is equal to

$$W_{\mathbf{1}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}^m)=C_{\mathbf{n}}(\boldsymbol{\alpha},\mathbf{D}^m)B_{\mathbf{n}}^{(l,m)}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}^m)|\mathbf{D}^m). \tag{26}$$

The partial polynomial part for the $\mathbf{j}$-modified matrix $\mathbf{D}_{\mathbf{j}}^m$ reads

$$W_{\mathbf{1}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}_{\mathbf{j}}^m)=C_{\mathbf{n}}(\boldsymbol{\alpha},\mathbf{D}_{\mathbf{j}}^m)B_{\mathbf{n}}^{(l,m)}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}_{\mathbf{j}}^m)|\mathbf{D}_{\mathbf{j}}^m),$$

and similarly to the scalar case the partial vector Sylvester wave $W_{\mathbf{j}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}^m)$ for arbitrary $\mathbf{j}\neq\mathbf{1}$ can be written as a linear superposition of the partial polynomial part for the matrix $\mathbf{D}_{\mathbf{j}}^m$ multiplied by the corresponding prime circulator:

$$W_{\mathbf{j}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}^m)=\sum_{r_i=0}^{p_i-1} W_{\mathbf{1}}^{\mathbf{n}}(\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}},\mathbf{D}_{\mathbf{j}}^m)\Psi_{\mathbf{j}}(\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}). \tag{27}$$

Combining (23,25) and (27) we arrive at the final expression of the restricted vector partition function as a linear superposition of the vector Bernoulli polynomials of higher order multiplied by the vector prime circulators:

$$W(\mathbf{s},\mathbf{D}^m)=\sum_{\mathbf{j}}\sum_{|\mathbf{n}|=m-l}\sum_{r_i=0}^{p_i-1} W_{\mathbf{1}}^{\mathbf{n}}(\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}},\mathbf{D}_{\mathbf{j}}^m)\Psi_{\mathbf{j}}(\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}), \tag{28}$$

where $W_{\mathbf{1}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}_{\mathbf{j}}^m)$ is given by

$$W_{\mathbf{1}}^{\mathbf{n}}(\mathbf{s},\mathbf{D}_{\mathbf{j}}^m)=\frac{\boldsymbol{\alpha}^{\mathbf{n}+\mathbf{1}}}{\mathbf{n}!P_m(\boldsymbol{\alpha},\mathbf{D}_{\mathbf{j}}^m)}\,B_{\mathbf{n}}^{(l,m)}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}_{\mathbf{j}}^m)|\mathbf{D}_{\mathbf{j}}^m). \tag{29}$$

It also can be written in the form of a mixture of partial waves

$$W(\mathbf{s},\mathbf{D}^m)=\sum_{|\mathbf{n}|=m-l} C_{\mathbf{n}}(\boldsymbol{\alpha},\mathbf{D}^m)W^{\mathbf{n}}(\mathbf{s},\mathbf{D}^m), \tag{30}$$

where the superposition coefficients depend on the vector $\boldsymbol{\alpha}$ and the partial wave is given by

$$W^{\mathbf{n}}(\mathbf{s},\mathbf{D}^m)=\sum_{\mathbf{j}}\sum_{r_i=0}^{p_i-1}\pi^{-1}(\mathbf{p}^{m-\omega_{\mathbf{j}}})B_{\mathbf{n}}^{(l,m)}(\mathbf{s}+\boldsymbol{\sigma}(\mathbf{D}_{\mathbf{j}}^m)-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}|\mathbf{D}_{\mathbf{j}}^m)\Psi_{\mathbf{j}}(\mathbf{s}-\mathbf{r}\cdot\mathbf{D}^{m-\omega_{\mathbf{j}}}).$$

In the scalar case we have $l=1$, $\mathbf{D}^m=\mathbf{d}^m, n=m-1$, and $P_m(\alpha,\mathbf{d}_j^m)=\pi(\mathbf{d}_j^m)\alpha^m$. Thus, the expression (29) reduces to (18), and noting that in the scalar case all $p_i=j$, one finds that (27) transforms into (19).

We present several examples of application of the formula (28) and show that it gives not only the vector partition function but also enables to find the chamber structure of the system.

4.1 Example 1

Consider computation of the vector partition function for the matrix $(m=3, l=2)$

$$\mathbf{D}^3=\left(\begin{array}{ccc}1&2&0\\1&0&1\end{array}\right),\qquad \mathbf{s}=(s_1,s_2),$$

first using the partial fraction expansion suggested in [2]. An idea of the method is based on the definition of the generating function (20). It is easily seen that $W(\mathbf{s},\mathbf{D})$ can be computed as the constant term in the expansion of the following expression:

$$W(\mathbf{s},\mathbf{D}^m)=\text{const}\left[\frac{1}{\mathbf{t}^{\mathbf{s}}}\prod_{i=1}^{m}\frac{1}{1-\mathbf{t}^{\mathbf{c}_i}}\right]. \tag{31}$$

Assuming all but one (say t_1) components of the vector $\mathbf{t}$ to be constant expand the r.h.s. in (31) into the partial fractions in t_1. This expansion contains both analytic and meromorphic parts w.r.t $t_1=0$. The meromorphic part doesn't contribute to the t_1-constant term, so it can be dropped. The constant term of the analytic part depends on the remaining components of $\mathbf{t}$, so that such expansion eliminates t_1. Applying this procedure repeatedly we eliminate all components of $\mathbf{t}$ and arrive to the result.

Assume t_1 constant and make partial fraction expansion w.r.t. t_2, obtaining

$$\frac{1}{(1-t_2)(1-t_1t_2)t_2^{s_2}}=\frac{1}{1-t_1}\left[\frac{1}{1-t_2}-\frac{t_1^{s_2+1}}{1-t_1t_2}\right]+\text{MMP},$$

where MMP stands for the meromorphic part. Thus we have

$$\begin{aligned}W(\mathbf{s},\mathbf{D}^3)&=\text{const}_{t_1}\left[\frac{1}{(1-t_1^2)(1-t_1)t_1^{s_1}}\text{const}_{t_2}\left[\frac{1}{1-t_2}-\frac{t_1^{s_2+1}}{1-t_1t_2}\right]\right]\\&=\text{const}_t\left[\frac{1-t^{s_2+1}}{(1-t^2)(1-t)t^{s_1}}\right].\end{aligned}$$

Finding that for $a\geq 0$

$$\text{const}_t\frac{1}{(1-t^2)(1-t)t^a}=\text{const}_t\left[\frac{1}{2(1-t)^2}+\frac{1+2a}{4(1-t)}+\frac{(-1)^a}{4(1+t)}+\text{MMP}\right]=\frac{a}{2}+\frac{3+(-1)^a}{4},$$

we obtain (note that $s_1\geq 0$)

$$\text{const}_t\left[\frac{1}{(1-t^2)(1-t)t^{s_1}}\right]=\frac{s_1}{2}+\frac{3+(-1)^{s_1}}{4},$$

and

$$\mathrm{const}_t\left[\frac{1}{(1-t^2)(1-t)t^{s_1-s_2-1}}\right]=\begin{cases}0, & s_1-s_2-1<0,\\ \frac{s_1-s_2}{2}+\frac{1-(-1)^{s_1-s_2}}{4}, & s_1-s_2-1\geq 0.\end{cases}$$

Combining these results we arrive at the final expression for the partition function

$$W(\mathbf{s},\mathbf{D}^3)=\begin{cases}\frac{s_1}{2}+\frac{3+(-1)^{s_1}}{4}, & s_1-s_2-1<0,\\ \frac{s_2+1}{2}+\frac{(-1)^{s_1}+(-1)^{s_1-s_2}}{4}, & s_1-s_2-1\geq 0.\end{cases} \tag{32}$$

It is easy to check that the expressions in (32) coincide for $s_1=s_2$ and $s_1=s_2-1$.

Turning to the formula (30) and having $|\mathbf{n}|=1$, we find that there are two partial waves corresponding to $\mathbf{n}_1=(1,0),\mathbf{n}_2=(0,1)$. Computing the polynomial $P_3(\boldsymbol{\alpha},\mathbf{D}^3)=2\alpha_1\alpha_2(\alpha_1+\alpha_2)$ we obtain the coefficients:

$$C_{(1,0)}(\boldsymbol{\alpha},\mathbf{D}^3)=\frac{\alpha_1}{2(\alpha_1+\alpha_2)},\quad C_{(0,1)}(\boldsymbol{\alpha},\mathbf{D}^3)=\frac{\alpha_2}{2(\alpha_1+\alpha_2)}.$$

Finding $\boldsymbol{\sigma}(\mathbf{D}^3)=(3,2)$ and using (26) we have for the partial polynomial parts

$$W_{\mathbf{1}}^{(1,0)}(\mathbf{s},\mathbf{D}^3)=C_{(1,0)}(\boldsymbol{\alpha},\mathbf{D}^3)\left(s_1+\frac{3}{2}\right),\quad W_{\mathbf{1}}^{(0,1)}(\mathbf{s},\mathbf{D}^3)=C_{(0,1)}(\boldsymbol{\alpha},\mathbf{D}^3)(s_2+1). \tag{33}$$

In addition to the polynomial contributions there are nonzero terms corresponding to $\mathbf{j}_1=(2,1)$ and $\mathbf{j}_2=(2,2)$, while the term with $\mathbf{j}=(1,2)$ doesn't contribute to the final result.

The $\mathbf{j}_1$-modified matrix $\mathbf{D}^m_{\mathbf{j}_1}$ reads ($p_3=2$)

$$\mathbf{D}^3_{(2,1)}=\left(\begin{array}{ccc}2&0&2\\0&1&2\end{array}\right),$$

where the columns are sorted already. In order to use (27) we first compute the partial polynomials for the $\mathbf{j}_1$-modified matrix:

$$W_{\mathbf{1}}^{(1,0)}(\mathbf{s},\mathbf{D}^3_{(2,1)})=C_{(1,0)}(\boldsymbol{\alpha},\mathbf{D}^3)\left(\frac{s_1}{2}+1\right),\quad W_{\mathbf{1}}^{(0,1)}(\mathbf{s},\mathbf{D}^3_{(2,1)})=C_{(0,1)}(\boldsymbol{\alpha},\mathbf{D}^3)\left(\frac{s_2}{2}+\frac{3}{4}\right),$$

and find

$$W_{(2,1)}^{\mathbf{n}_i}(\mathbf{s},\mathbf{D}^3)=\sum_{r=0}^{1}W_{\mathbf{1}}^{\mathbf{n}_i}((s_1-r,s_2-r),\mathbf{D}^3_{(2,1)})\Psi_2(s_1-r),$$

arriving at

$$W_{(2,1)}^{(1,0)}(\mathbf{s},\mathbf{D}^3)=\frac{1}{2}C_{(1,0)}(\boldsymbol{\alpha},\mathbf{D}^3)\Psi_2(s_1),\quad W_{(2,1)}^{(0,1)}(\mathbf{s},\mathbf{D}^3)=\frac{1}{2}C_{(0,1)}(\boldsymbol{\alpha},\mathbf{D}^3)\Psi_2(s_1). \tag{34}$$

The $\mathbf{j}_2$-modified matrix $\mathbf{D}^m_{\mathbf{j}_2}$ reads ($p_2=p_3=2$)

$$\mathbf{D}^3_{(2,2)}=\left(\begin{array}{ccc}2&0&2\\0&2&2\end{array}\right).$$

Repeating the computation we start with the partial polynomials for the $\mathbf{j}_2$-modified matrix:

$$W_{\mathbf{1}}^{(1,0)}(\mathbf{s},\mathbf{D}^3_{(2,2)})=C_{(1,0)}(\boldsymbol{\alpha},\mathbf{D}^3)\left(\frac{s_1}{4}+\frac{1}{2}\right),\quad W_{\mathbf{1}}^{(0,1)}(\mathbf{s},\mathbf{D}^3_{(2,2)})=C_{(0,1)}(\boldsymbol{\alpha},\mathbf{D}^3)\left(\frac{s_2}{4}+\frac{1}{2}\right).$$

The formula (27) takes form

$$W^{\mathbf{n}_i}_{(2,2)}(\mathbf{s},\mathbf{D}^3)=\sum_{r_1,r_2=0}^{1} W^{\mathbf{n}_i}_{\mathbf{1}}((s_1-r_1,s_2-r_1-r_2),\mathbf{D}^3_{(2,2)})\Psi_2(s_1-r_1)\Psi_2(s_2-r_1-r_2),$$

and we obtain

$$W^{(1,0)}_{(2,2)}(\mathbf{s},\mathbf{D}^3)=0,\quad W^{(0,1)}_{(2,2)}(\mathbf{s},\mathbf{D}^3)=\frac{1}{2}C_{(0,1)}(\boldsymbol{\alpha},\mathbf{D}^3)\Psi_2(s_1)\Psi_2(s_2). \tag{35}$$

Combining the expressions (33), (34) and (35) we arrive at the final result

$$W(\mathbf{s},\mathbf{D}^3)=C_{(1,0)}(\boldsymbol{\alpha},\mathbf{D}^3)W^{(1,0)}(\mathbf{s},\mathbf{D}^3)+C_{(0,1)}(\boldsymbol{\alpha},\mathbf{D}^3)W^{(0,1)}(\mathbf{s},\mathbf{D}^3), \tag{36}$$

where

$$\begin{aligned} W^{(1,0)}(\mathbf{s},\mathbf{D}^3) &= s_1+\frac{3}{2}+\frac{1}{2}\Psi_2(s_1),\\ W^{(0,1)}(\mathbf{s},\mathbf{D}^3) &= s_2+1+\frac{1}{2}\Psi_2(s_1)+\frac{1}{2}\Psi_2(s_1-s_2). \end{aligned}$$

Noting that $\Psi_2(s)=(-1)^s$ one can check by straightforward computation that the case $\alpha_2/\alpha_1=0$ produces the first line in (32), while $\alpha_1/\alpha_2=0$ gives the last line. We see that proper choice of the vector $\boldsymbol{\alpha}$ corresponds to the selection of the chamber.

4.2 Example 2

Consider another example for $l=2, m=4$ and

$$\mathbf{D}^4=\left(\begin{array}{cccc}1&2&1&0\\1&1&0&1\end{array}\right),\qquad \mathbf{s}=(s_1,s_2).$$

The straightforward computation using the partial fraction expansion gives the following result (see [2]):

$$W(\mathbf{s},\mathbf{D}^4)=\begin{cases}\frac{s_1^2}{4}+s_1+\frac{7+(-1)^{s_1}}{8}, & s_1\le s_2,\\ s_1s_2-\frac{s_1^2+2s_2^2}{4}+\frac{s_1+s_2}{2}+\frac{7+(-1)^{s_1}}{8}, & s_1/2-1\le s_2\le s_1+1,\\ \frac{s_2^2}{2}+\frac{3s_2}{2}+1, & s_2\le s_1/2.\end{cases} \tag{37}$$

As $|\mathbf{n}|=2$, we have three partial waves corresponding to $\mathbf{n}_1=(2,0), \mathbf{n}_2=(1,1), \mathbf{n}_3=(0,2)$. Computing the polynomial $P_4(\boldsymbol{\alpha},\mathbf{D}^4)=\alpha_1\alpha_2(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)$ we find the coefficients:

$$C_{(2,0)}(\boldsymbol{\alpha},\mathbf{D}^4)=\frac{\alpha_1^2}{2(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)},$$

$$C_{(1,1)}(\boldsymbol{\alpha},\mathbf{D}^4)=\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)},$$

$$C_{(0,2)}(\boldsymbol{\alpha},\mathbf{D}^4)=\frac{\alpha_2^2}{2(\alpha_1+\alpha_2)(2\alpha_1+\alpha_2)}.$$

Finding $\boldsymbol{\sigma}(\mathbf{D}^4)=(4,3)$ and using (26) we have for the partial polynomial parts

$$\begin{aligned} W^{(2,0)}_{\mathbf{1}}(\mathbf{s},\mathbf{D}^4) &= C_{(2,0)}(\boldsymbol{\alpha},\mathbf{D}^4)\left(s_1^2+4s_1+\frac{7}{2}\right),\\ W^{(1,1)}_{\mathbf{1}}(\mathbf{s},\mathbf{D}^4) &= C_{(1,1)}(\boldsymbol{\alpha},\mathbf{D}^4)\left(s_1s_2+\frac{3s_1}{2}+2s_2+\frac{11}{4}\right),\\ W^{(0,2)}_{\mathbf{1}}(\mathbf{s},\mathbf{D}^4) &= C_{(0,2)}(\boldsymbol{\alpha},\mathbf{D}^4)\left(s_2^2+3s_2+2\right). \end{aligned} \tag{38}$$

In addition to the polynomial part only the term corresponding to $\mathbf{j} = (2,1)$ produces nonzero contribution to the vector partition function, while two other terms with $\mathbf{j} = (1,2)$ and $\mathbf{j} = (2,2)$ don't contribute into the result. The $\mathbf{j}$-modified matrix $\mathbf{D}_{\mathbf{j}}^m$ reads ($p_3 = p_4 = 2$)

$$\mathbf{D}^4_{(2,1)} = \left(\begin{array}{cccc} 2 & 0 & 2 & 2 \\ 1 & 1 & 2 & 0 \end{array}\right).$$

We first compute the partial polynomials for the $\mathbf{j}$-modified matrix:

$$\begin{aligned} W_{\mathbf{1}}^{(2,0)}(\mathbf{s}, \mathbf{D}^4_{(2,1)}) &= C_{(2,0)}(\boldsymbol{\alpha}, \mathbf{D}^4) \left(\frac{s_1^2}{4} + \frac{3s_1}{2} + 2\right), \\ W_{\mathbf{1}}^{(1,1)}(\mathbf{s}, \mathbf{D}^4_{(2,1)}) &= C_{(1,1)}(\boldsymbol{\alpha}, \mathbf{D}^4) \left(\frac{s_1 s_2}{4} + \frac{s_1}{2} + \frac{3s_2}{4} + \frac{11}{8}\right), \\ W_{\mathbf{1}}^{(0,2)}(\mathbf{s}, \mathbf{D}^4_{(2,1)}) &= C_{(0,2)}(\boldsymbol{\alpha}, \mathbf{D}^4) \left(\frac{s_2^2}{4} + s_2 + \frac{7}{8}\right), \end{aligned}$$

and use (27) to have

$$W_{(2,1)}^{\mathbf{n}_i}(\mathbf{s}, \mathbf{D}^4) = \sum_{r_3=0}^{1}\sum_{r_4=0}^{1} W_{\mathbf{1}}^{\mathbf{n}_i}((s_1 - r_3 - r_4, s_2 - r_3), \mathbf{D}^4_{(2,1)})\Psi_2(s_1 - r_3 - r_4),$$

and obtain

$$\begin{aligned} W_{(2,1)}^{(2,0)}(\mathbf{s}, \mathbf{D}^4) &= \frac{1}{2} C_{(2,0)}(\boldsymbol{\alpha}, \mathbf{D}^4)\Psi_2(s_1), \\ W_{(2,1)}^{(1,1)}(\mathbf{s}, \mathbf{D}^4) &= \frac{1}{4} C_{(1,1)}(\boldsymbol{\alpha}, \mathbf{D}^4)\Psi_2(s_1), \\ W_{(2,1)}^{(0,2)}(\mathbf{s}, \mathbf{D}^4) &= 0. \end{aligned} \tag{39}$$

Combining the expressions (38) and (39) we arrive at the final result

$$W(\mathbf{s}, \mathbf{D}^4) = C_{(2,0)}(\boldsymbol{\alpha}, \mathbf{D}^4) W^{(2,0)}(\mathbf{s}, \mathbf{D}^4) + C_{(1,1)}(\boldsymbol{\alpha}, \mathbf{D}^4) W^{(1,1)}(\mathbf{s}, \mathbf{D}^4) + C_{(0,2)}(\boldsymbol{\alpha}, \mathbf{D}^4) W^{(0,2)}(\mathbf{s}, \mathbf{D}^4),$$

with

$$\begin{aligned} W^{(2,0)}(\mathbf{s}, \mathbf{D}^4) &= s_1^2 + 4s_1 + \frac{7}{2} + \frac{1}{2}\Psi_2(s_1), \\ W^{(1,1)}(\mathbf{s}, \mathbf{D}^4) &= s_1 s_2 + \frac{3s_1}{2} + 2s_2 + \frac{11}{4} + \frac{1}{4}\Psi_2(s_1), \\ W^{(0,2)}(\mathbf{s}, \mathbf{D}^4) &= s_2^2 + 3s_2 + 2. \end{aligned}$$

We find that the case $\alpha_2/\alpha_1 = 0$ produces the first line in (37), while $\alpha_1/\alpha_2 = 0$ corresponds to the last line. The second line is obtained as a real part of $W(\mathbf{s}, \mathbf{D}^4)$ for $\alpha_2/\alpha_1 = -1 \pm i$. Thus, we see again that the choice of the vector $\boldsymbol{\alpha}$ corresponds to selection of the chamber.

4.3 Example 3

It should be noted that the formula (28) remains valid also in the case $m = l$. To illustrate it consider the case $m = l = 2$ with

$$\mathbf{D}^2 = \left(\begin{array}{cc} 1 & 2 \\ 1 & 0 \end{array}\right), \qquad \mathbf{s} = (s_1, s_2).$$

The solution of the corresponding linear problem reads $x_1 = s_2,\ x_2 = (s_1 - s_2)/2$, thus the number of solutions equals to one for even nonnegative differences $s_1 - s_2$ and zero otherwise. This can be written as

$$W(\mathbf{s}, \mathbf{D}^2) = \begin{cases} [1 + (-1)^{s_1 - s_2}]/2, & s_1 \geq s_2, \\ 0, & s_1 < s_2. \end{cases} \tag{40}$$

As $m = l$ we have $\mathbf{n} = \mathbf{0} = (0, 0)$, and all $B_{\mathbf{0}}^{(m,m)}(\mathbf{s}|\mathbf{D}_{\mathbf{j}}^m) = 1$. The polynomial part is found as

$$W_{\mathbf{1}}^{\mathbf{0}}(\mathbf{s}, \mathbf{D}^2) = \frac{\alpha_2}{2(\alpha_1 + \alpha_2)}. \tag{41}$$

In addition to the polynomial part only the term corresponding to $\mathbf{j} = (2, 2)$ produces nonzero contribution to the vector partition function, while two other terms with $\mathbf{j} = (1, 2)$ and $\mathbf{j} = (2, 1)$ don't contribute into the result. The $\mathbf{j}$-modified matrix $\mathbf{D}_{\mathbf{j}}^m$ reads $(p = 2)$

$$\mathbf{D}_{(2,2)}^2 = \left(\begin{array}{c} 2\ 2 \\ 0\ 2 \end{array} \right),$$

and the corresponding term is found

$$W_{(2,2)}^{\mathbf{0}}(\mathbf{s}, \mathbf{D}_{(2,2)}^2) = \frac{\alpha_2}{4(\alpha_1 + \alpha_2)} \sum_{r=0}^{1} \Psi_2(s_1 - r)\Psi_2(s_2 - r) = \frac{\alpha_2}{2(\alpha_1 + \alpha_2)}(-1)^{s_1 - s_2}. \tag{42}$$

Combining (41) and (42) we arrive at the expression

$$W(\mathbf{s}, \mathbf{D}^2) = \frac{\alpha_2}{2(\alpha_1 + \alpha_2)}[1 + (-1)^{s_1 - s_2}].$$

This result also can be obtained from (36) applying to it the recursive relation (21) with $\mathbf{c}_3 = (0, 1)$. It is easy to check that the choice $\alpha_1/\alpha_2 = 0$ corresponds to the first line in (40), while $\alpha_2/\alpha_1 = 0$ produces the second one.

4.4 Example 4

Finally consider the case $m = 4, l = 3$ with

$$\mathbf{D}^4 = \left(\begin{array}{c} 2\ 1\ 0\ 0 \\ 0\ 1\ 1\ 2 \\ 0\ 0\ 1\ 0 \end{array} \right), \qquad \mathbf{s} = (s_1, s_2, s_3).$$

The straightforward computation using the partial fraction expansion gives the following result

$$W(\mathbf{s}, \mathbf{D}^4) = \begin{cases} (1 + (-1)^{s_1 + s_2 + s_3}) \left(\frac{3 + (-1)^{s_1}}{8} + \frac{s_1}{4} \right), & s_2 \geq s_3,\ s_1 - s_2 + s_3 - 2 < 0, \\ (1 + (-1)^{s_1 + s_2 + s_3}) \left(\frac{3 + (-1)^{s_1}}{8} + \frac{s_2 - s_3}{4} \right), & s_2 \geq s_3,\ s_1 - s_2 + s_3 - 2 \geq 0, \\ 0, & s_2 < s_3. \end{cases} \tag{43}$$

There are three partial waves corresponding to $\mathbf{n}_1 = (1, 0, 0),\ \mathbf{n}_2 = (0, 1, 0),\ \mathbf{n}_1 = (0, 0, 1)$. We drop the computation details and present the final result in the form

$$W(\mathbf{s}, \mathbf{D}^4) = \frac{\alpha_3[1 + (-1)^{s_1 + s_2 + s_3}]}{8(\alpha_1 + \alpha_2)(\alpha_2 + \alpha_3)}[3\alpha_1 + 4\alpha_2 + (-1)^{s_1}(\alpha_1 + \alpha_2) + (-1)^{s_3}(\alpha_2 + \alpha_3) + 2\boldsymbol{\alpha} \cdot \mathbf{s}].$$

The comparison of the real part of the above expression with (43) leads to the following values of the vector $\boldsymbol{\alpha}$ – the first line in (43) corresponds to $\alpha_2/\alpha_1 = 0, \alpha_3/\alpha_1 = \pm i$, the second one is obtained with $\alpha_2/\alpha_1 = \pm i, \alpha_3/\alpha_1 = 1 \mp i$, and the last line is given by $\alpha_3/\alpha_1 = \alpha_3/\alpha_2 = 0$.

4.5 Discussion

The formula (28) provides an exact solution of the vector restricted partition problem through the vector Bernoulli polynomials of higher order. It also can be written in the form of a mixture of partial waves (30) and the superposition coefficients depend on the vector $\boldsymbol{\alpha}$. It appears that proper choice of this vector parameter produces the solution in one of the system chambers. The procedure for selection of the vector $\boldsymbol{\alpha}$ will be published elsewhere.

Knowledge of the vector $\boldsymbol{\alpha}$ for each chamber gives way to determine chamber boundaries. The partition function (as well as its Sylvester waves) in two adjacent chambers should coincide at their boundary. Thus, setting the corresponding values to $\boldsymbol{\alpha}$ and equating the real part of resulting expressions for $W(\mathbf{s}, \mathbf{D})$ one obtains the linear relations between components of the vector $\mathbf{s}$ which determines the location of the boundary.

Thus, the formulas (28) and (30) appear to be the most general as they not only give the expression for the vector partition function in every chamber of the system, but also determine shape of each individual chamber.

The derivation of (28) was made in assumption that the matrix $\mathbf{D}$ is non-degenerate. The degeneracy of the matrix related to linear dependence of its columns doesn't affect the solution of the problem. It happens only if the matrix rows are linearly dependent. This dependence leads to effective reduction of the row number l and to additional linear conditions imposed on the components of the vector $\mathbf{s}$. These conditions should be satisfied to have nonvanishing value of the vector partition function. This way the original problem is reduced to the non-degenerate case.

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