On automated derivation of amplitude equations in nonlinear bifurcation problems

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Abstract

Examples of computer-assisted derivation of the amplitude equations valid in the vicinity of a bifurcation point are presented. The methodology created using *Mathematica* computer software language is outlined, so that its potential to tackle a wide class of problems in nonlinear sciences appears.

1 Introduction

Problems of fluid mechanics, transport phenomena, elasticity theory, electrodynamics of continua, nonlinear optics in material media (including laser dynamics) lead, as a rule, to complex sets of nonlinear PDEs that are usually amenable to numerical treatment only, and often exhaust the abilities of even fastest supercomputers. In many cases of interest, one can obtain rational approximations to these complex systems that still retain most important features of their dynamic behavior. In problems of lubrication theory, mechanics of rods and shells, optical transmission in wave guides, and others, this can be done by exploiting the disparity of scaling in different directions that is a consequence of special geometry of these systems. In other cases, the disparity of scaling has a more subtle origin, as it may be related to extreme values of certain parameters of the problem, like, say, Reynolds or Prandtl numbers or a low heat transport coefficient.

Another situation where behavior of systems of different physical origin is described by representative equations (belonging to a universal class) arises under conditions close to "critical" behaviour, *i.e.*, in the vicinity of threshold values of parameters that mark a point of qualitative change in behavior, like in transition from a quiescent state to convection, or from rigid mechanical equilibrium to vibration, or from a homogenious to patterned state of a material medium.

There is a number of celebrated "universal" equations which have been extensively studied in physical literature. Some of them were first suggested as models, others were derived as rational approximations in a certain physical context, and then discovered again in apparently unrelated problems. The general structure of these equations can be often predicted by symmetry considerations. However, rational derivation required for establishing quantitative relations between their coefficients and parameters of underlying physical systems has been carried out in scientific literature on a case-to-case basis. In general, manipulations leading to such equations are repetitive and time-consuming. Although the general method is well established, application to particular problems still remains an advanced research topic requiring great effort and sometimes abandoned just because it is practically impossible to obtain reliable result in a long chain of computations done by hand or even with the help of a machine-assisted but non-systematic procedure.

It is important to note that, except simplest cases when coefficients can be removed by rescaling, different qualitative features of the behavior are observed in different parameter ranges. When relations between coefficients of universal equations and measurable physical parameters are known, the domains determined by studying universal equations can be mapped upon the actual parameter space, and then further continued into a region where universal equations are no longer valid quantitatively but still faithfully describe qualitative features of behavior of a complex nonlinear system.

Analytical or computer studies often use two or more consecutive approximations dependent on presence of different small parameters, *e.g.*, a scale ratio δ and a deviation from a critical point ϵ. Generally, different limits (*e.g.*, $\delta \to 0$ and $\epsilon \to 0$) do not commute [Knobloch, 1995], and, since one is really interested in the behavior at finite values of a small parameter, continuing the expansion to higher orders becomes necessary. Higher orders must be also taken into account in cases when an approximate symmetry is retained in the lowest order but broken in higher orders, as it happens, *e.g.*, in weakly dissipative systems. Calculations beyond the leading order are forbiddingly heavy, and therefore there are only very few studies of systems of this kind.

In order to perform the above mentioned calculations more consistently and faster a

set of the programs implementing well established algorithms, both symbolically and numerically, have been developed using the *Mathematica* symbolic algebra language [Wolfram, 1991]. *Mathematica* is widely known as the most sophisticated and, at the same time, most commercially successful general-purpose mathematical software system combining logical, analytical, numerical and graphical functions. Unlike other common symbolic and numerical software systems, *Mathematica* is also a *programming language* that gives convenient tools for expanding its capabilities. Two programs of this set, `BifurcationCondition` and `Unfolding` were used for the derivation of amplitude equations in a paradigmatic example (Section 2). A short description of these programs is given in Section 3.

2 2D dissipation–modified KdV equation

The convection in a liquid layer open to air and heated from the air side (Marangoni – Bénard convection) may lead to richness of dynamical behaviour including excitation of transverse and longitudal waves, and solitary waves [Chu & Velarde, 1988; Nepomnyashchy & Velarde, 1994; Nekorkin & Velarde, 1994; Velarde *et al.*, 1995]. The description of the dynamics is given by an evolution equation which is the natural generalization of the Korteweg – de Vries (KdV), Kadomtsev – Petriashvili (KP) and Boussinesq theories [See *e.g.*, Drazin & Johnson, 1989]. Similar equations also appear when dissipation is introduced in the study of internal waves [Zimmerman & Velarde, 1994] and electronic circuits [Nekorkin *et al.*, 1994].

In the Marangoni-Bénard instability the equation is

$$u_t + b_1 u u_x + b_2 u_{xxx} + b_3 u_{xx} + b_4 u_{xxxx} + b_5 (u u_x)_x = 0, \tag{1}$$

where $u(x,t)$ denotes surface deformation. Coefficients b_i $(i = 1, \ldots, 5)$ are assumed to be real. Particular values of the coefficients in the Eq.(1) yield the KdV equation ($b_3 = b_4 = b_5 = 0$), the Burgers equation ($b_2 = b_4 = b_5 = 0$), indeed a combination of both ($b_4 = b_5 = 0$) [Drazin & Johnson, 1989], the Kuramoto-Sivashinsky (KS) equation ($b_2 = b_5 = 0$) [Shkadov, 1967; Nepomnyashchy, 1974; Kuramoto & Tsuzuki, 1975; Sivashinsky, 1976] or to extended KS equation ($b_2 = 0$) [Hyman & Nicolaenko, 1987].

In the moving frame with new coordinate $\xi = x + c_0 t$ (c_0 denotes the phase veocity or wave speed of the expected wave; the choice of a left-moving wave rather than a right-moving one is immaterial for our purpose here) the Eq.(1) may be rewritten as a set of three nonlinear ordinary differential equations, *i.e.*, as a dynamical system

$$\begin{aligned} \dot{u} &= y, \\ \dot{y} &= z, \\ \gamma \dot{z} &= -\beta z - \alpha u y - \nu y - F(u), \end{aligned} \tag{2}$$

where the dot denotes differentiation with respect to ξ, $F(u) = cu + u^2/2$ and $c = c_0/b_1, \beta = b_2/b_1, \nu = b_3/b_1, \gamma = b_4/b_1$ and $\alpha = b_5/b_1$. For the sake of symmetry the following transformations were applied to the above system $u \to u - c, \nu \to \nu + \alpha c$ leading to the new function $F(u) = (u^2 - c^2)/2$.

2.1 Stationary bifurcation

The system posesses a stationary bifurcation point located at the origin $\{u, y, z\} = \{0, 0, 0\}$ when the phase velocity is equal to zero ($c = 0$).

In[1]:=
bp = Apply[Solve,
BifurcationCondition[{y,z,(c^2/2 - u^2/2 - nu y -
alpha u y - beta z)/gamma,{u,y,z},{c}]] // First
Out[1]=
{z → 0, c → 0, u → 0, y → 0}.
It appears that the bifurcation occurs at arbitrary values of the parameters $\alpha, \beta, \nu, \gamma$.

The amplitude equations valid in the vicinity of the origin produced by the function `Unfolding` read:

$$\partial_1 a = (c_1^2 - a^2)/2\nu, \partial_2 a = (\beta - \alpha\nu)a(c_1^2 - a^2)/2\nu^3, \tag{3}$$

where a denotes the (small) amplitude of the perturbation, and c_1 are the first-order perturbation of the bifurcation parameter c.

2.2 Hopf bifurcation

A more difficult task is the derivation of the amplitude equation in the vicinity of a Hopf bifurcation point. The location of such point may be determined using the function `BifurcationCondition` with the option value `Special → Hopf`.
In[2]:=
bph = Apply[Solve,
BifurcationCondition[{y,z,(c^2/2 - u^2/2 - nu y -
alpha u y - beta z)/gamma,{u,y,z},{gamma},
Special → Hopf]]
Out[2]=
{{z → 0, gamma → beta (alpha c - nu)/c, y → 0, u → -c},
{z → 0, gamma → beta (alpha c + nu)/c, y → 0, u → c} }

The system (3) consideration has two Hopf bifurcation points $P_\pm = \{\pm c, 0, 0\}$ located symmetrically with recpect to the origin at the corresponding values of the bifurcation parameter $\gamma_\pm = \alpha\beta \pm \nu\beta/c$, where the limit cycles with a frequencies $\omega_\pm = \sqrt{\pm c/\beta}$ will appear. This fact simply confirms the existing symmetry with respect to the direction of wave motion (to the sign of phase velocity).

The amplitude equation derived by the function `Unfolding` with an option value `Special → Hopf` can be written in the form:

$$\partial_2 a = -\frac{(1 + i\alpha\omega_\pm)[12\gamma_\pm\omega_\pm - 5i\beta - 2\alpha\beta\omega_\pm]}{12\beta\omega_\pm^7[2(\gamma_\pm\omega_\pm)^2 - 3i\gamma_\pm\omega_\pm\beta - \beta^2]}|a|^2 a + \frac{\omega_\pm^2}{2(i\gamma_\pm\omega_\pm + \beta)}a\gamma_2, \tag{4}$$

where γ_2 is the second-order approximation of the bifurcation parameter γ from the critical value $\gamma_\pm$. Note that the amplitude a is assumed to be complex.

If the bifurcation point is numerically defined, the function `Unfolding` computes the amplitude equations also numerically. In this case it is possible to proceed to higher orders. For illustration we presenting the result of the derivation of the amplitude equations in the vicinity of the Hopf bifurcation point P_+ for $(\alpha = 0.2, \beta = 0.3, \nu = 1.2, c = 0.5)$:

$$\begin{aligned}\partial_2 a &= (-0.227817 - 0.144426\,i)\,|a|^2 a + (0.226449 - 0.760096\,i)\,a\gamma_2,\\ \partial_4 a &= (-0.0671936 + 0.0253176\,i)\,|a|^4 a + (0.378532 + 0.171444\,i)\,|a|^2 a\gamma_2 +\\ &\qquad (-0.310128 + 0.717909\,i)\,a\gamma_2^2.\end{aligned} \tag{5}$$

3 Short Description of the *Mathematica* functions

To make this paper almost selfcontained in this section the various functions of *Mathematica* earlier mentioned are given. The description includes the definitions of the arguments and options, and sketches the main steps of the algorithm implemented by each function.

3.1 Function `BifurcationCondition`

`BifurcationCondition[func,vars,params,options]` produces a set of algebraic equations determining a location of the bifurcation point of the specified type.

The arguments of `BifurcationCondition` are:

- `func` = $\mathbf{f}(\mathbf{u},\mathbf{p})$ denotes an array of functions, representing a right hand side of ordinary nonlinear differential equations;
- `vars` = $\mathbf{u}$ denotes an array of variables;
- `params` = $\mathbf{p}$ denotes an array of bifurcation parameters.

The type of the bifurcation point is specified by the option `Special` with *default* value `Special -> None` for the point of monotonic bifurcation. The other option value `Special -> Hopf` is used to obtain equations for the Hopf bifurcation point. This value also requires that length of the array `vars` be no less than two, and that there is a single bifurcation parameter.

`BifurcationCondition` implements consecutively the following subroutines:

- Checks the number i of variables `vars` and number q of parameters `params`; if both i and q are greater than unity the set of equations is not produced because it requires the determination of the eigenvectors of the problem.
- If $i = 1$ the set of equations is produced by equating to zero subsequent derivatives of the function f over u: $(f = 0, \quad df/du = 0, \quad d^2f/du^2 = 0, \ldots, d^qf/du^q = 0)$.
- If $i > 1$ there may be only one parameter ($q = 1$), and the set of equations is produced by adding the condition $\det(\partial\mathbf{f}/\partial\mathbf{u}) = 0$ to the original functions $\mathbf{f} = 0$. For a Hopf bifurcation this additional condition is replaced by the condition on the corresponding Hurwitz determinant.

3.2 Function `Unfolding`

`Unfolding[funs, var, bvar, param, bparam, amp, order, options]` derives a set of amplitude equations for a dynamical system $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u},\mathbf{p})$ in the vicinity of a bifurcation point of codimension `order`.

The arguments of `Unfolding` are:

- an array of functions `funs` = $\mathbf{f}(\mathbf{u},\mathbf{p})$, representing a right hand side of ordinary nonlinear differential equations;
- `vars` = $\mathbf{u}$ denotes an array of variables;
- `bvars` = $\mathbf{u}_0$ denotes an array of their values at the bifurcation point;

- `param` $= \mathbf{p}$ denotes an array of bifurcation parameters;
- `bparam` $= \mathbf{p}_0$ denotes an array of their values at the bifurcation point;
- `order` $= q$ denotes a codimension of the bifurcation manifold.

The only option of the function is `Special`. The *default* value `Special -> None` implies the generation of the amplitude equations in case of monotonic, stationary bifurcations. If other option value `Special -> Hopf` is used to obtain amplitude equations in the vicinity of the Hopf bifurcation point.

`Unfolding` sequentially implements the following subroutines:

- Generates equations of the bifurcation theory in a symbolic form determining subsequent *deviations* from the stationary (or periodic) solution calculated at the bifurcation point.
- Tests whether the bifurcation point provided is a stationary point of the system.
- Tests whether this point lies on the bifurcation manifold of codimension one. A Jacobian matrix $\mathbf{J} = \partial\mathbf{f}/\partial\mathbf{u}$ of the system is calculated at this point. Then the set of eigenvalues is computed and is examined on precence of the single zero (for monotonic bifurcations) or a pair of purely imaginary eigenvalues (for Hopf bifurcation). If the test is passed the values of the corresponding eigenvectors $\mathbf{U}, \mathbf{U}^\dagger$ of the Jacobian matrix and the transposed matrix are calculated.
- Finally, iteratively implements q times the procedure producing the set of amplitude equations.

4 Conclusion.

We have shown by the way of examples the usefulness of an automated computer assisted derivation ofamplitude equations in the vicinity of bifurcation points of nonlinear system implemented in the *Mathematica* computer language. The function `Bifurcation Condition` enables to determine the location of the bifurcation point of specified type (stationary or Hopf). When the bifurcation point is located (symbolically or numerically) the function `Unfolding` derives the amplitude equation valid in vicinity of the point.

Automatic algorithms for the analysis of nonlinear distributed systems provides basic tools saving time to researchers, and allowing comprehensive nonlinear analysis in cases when the required procedures are known in principle but cannot be practically implemented because of cumbersome and heavy computations involved.

As a conclusion we can say that the generalization of the tools illustrated in this paper allow to carry out the following operations:

- derivation of long-scale equations through dimensional reduction;
- construction of loci of symmetry-breaking bifurcations, using shooting or chasing routines for solution of the corresponding boundary value problems;
- construction of amplitude equations at stationary and oscillatory symmetry breaking bifurcations;

- construction of branches of spatially periodic solutions;
- basic analysis of phase dynamics and secondary instabilities;
- detection and analysis of degenerate bifurcations.

References

[1] Chu, X.-L. & Velarde, M.G. [1988] "Sustained transverse and longitudal waves at the open surface of a liquid," *Physicochem. Hydrodyn.* **10**, 727.

[2] Drazin, Ph. & Johnson, R.S. [1989] *"Solitons. An Introduction"*, Cambridge Univ. Press, Cambridge.

[3] Hyman, J.M. & Nicolaenko, B. [1987] "Cohernce and chaos in Kuramoto-Velarde equation," in *"Directions in Partial Differential Equations"*, ed. Crandall, M. (Academic Press, New York).

[4] Knobloch, E. [1995] in *"Nonlinear Dynamics and Pattern Formation in the Natural Environment"*, Pitman (Longman).

[5] Kuramoto, Y. & Tsuzuki, T. [1975] "On the formation of dissipative structures in reaction-diffusion systems," *Prog. Theoret. Phys.* **54**, 687.

[6] Nekorkin, V.I. & Velarde, M.G. [1994] "Solitary waves, soliton bound states and chaos in a dissipative Korteweg - de Vries equation," *Int. J. Bif. Chaos* **4**, 1135.

[7] Nekorkin, V.I., Kazantsev, V.B., Rulkov, N.F., Velarde, M.G., & Chua, L.O. [1995] "Homoclinic orbits and solitary waves in a one-dimensional array of Chua's circuits," *I.E.E.E.J. Trans. Circuits Syst.* **42**, 785.

[8] Nepomnyashchy, A.A. [1974] "On the stability of wavy motions in a film flowing down an inclined plane," *Mech. Zidk. Gaza* **3**, 28 (in Russian).

[9] Nepomnyashchy, A.A. & Velarde, M.G. [1994] "A three-dimensional description of solitary waves and their interaction in Marangoni-B'enard layers," *Phys.Fluids* **6**, 187.

[10] Shkadov, V.Ya. [1967] "Wave conditions in the flow of thin nlayer of a viscous liquid under the action of gravity," *Izv. Acad. Nauk. SSSR Mekh. Zhidk. i Gaza* **1**, 43 (in Russian).

[11] Sivashinsky, G.I. [1977] "Nonlinear analysis of hydrodynamic instability in laminar flames. I. Derivation of basic equations," *Acta Astron.* **4**, 1176.

[12] Velarde, M.G., Nekorkin, V.I. & Maksimov, A. [1995] "On further results on the evolution of solitary waves and their bound states of a dissipative Korteweg - de Vries equation," *Int. J. Bif. Chaos* **5**, 831.

[13] Wolfram, S. [1991] *"Mathematica. A System for Doing Mathematics by Computer"*, 2nd ed., Addison – Wesley.

[14] Zimmerman, W. & Velarde, M.G. [1994] "Nonlinear waves in stably stratified dissipative media. Solitary waves and turbulent bursts," *Phys. Scripta* **T55**, 111.