Multiple Liquid Bridges with Non-Smooth Interfaces

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Multiple liquid bridges with non-smooth interfaces

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Abstract. We consider a coexistence of two axisymmetric liquid bridges LB_i and LB_m of two immiscible liquids i and m which are immersed in a third liquid (or gas) e and trapped between two smooth solid bodies with axisymmetric surfaces S_1, S_2 and free contact lines. Evolution of liquid bridges allows two different configurations of LB_i and LB_m with multiple (five or three) interfaces of non-smooth shape. We formulate a variational problem with volume constraints and present its governing equations supplemented by boundary conditions. We find a universal relationship between curvature of the interfaces and discuss the Neumann triangle relations at the singular curve where all liquids meet together.

Mathematics Subject Classification. Primary 53A10; Secondary 76B45.

Keywords. Capillarity · Liquid bridge · Coexistence of three deformable interfaces.

1. Introduction

Consider an evolution of two liquid bridges LB_i and LB_m of immiscible liquids, i (inner) and m (intermediate), trapped between two axisymmetric smooth solid bodies with surfaces S_1, S_2 in such a way that LB_i is immersed into LB_m and the latter is immersed into the e (external) liquid (or gas) which occupies the rest of the space between the two bodies (see Fig. 1a). When liquid m begins to evaporate then LB_m reduces in volume (and width). Depending on the relationships between the contact angles of both liquids on S_1 and S_2, there are two scenarios for connectivity breakage of the liquid bridge m between the two solids. The first scenario (*five interfaces*) occurs when LB_m splits into two parts each supported by a different solid (see Fig. 1b). The second scenario (*three interfaces*) occurs when LB_m is left as a whole but has support only on the upper (see Fig. 2b) or lower solid.

Both scenarios lead to a new phenomenon which has not been discussed in literature before, namely an existence of multiple LBs with non-smooth interfaces. In contrast to the known LBs with fixed and free contact line (CL), here one of CLs appears as a line where three interfaces with different curvatures meet together. From a mathematical standpoint, this singular curve is governed by transversality conditions (in physics they are referred to as the Young relations), and coincidence conditions, i.e., three interfaces always intersect at one single curve. We derive a relationship combining the constant mean curvatures of three different interfaces and give the interfaces consistency rules for their coexistence. Another important result is the vectorial Neumann triangle relation at the triple point which is located on a singular curve.

2. Variational problem for five interfaces

Consider a functional $E[r, z]$ of surface energy

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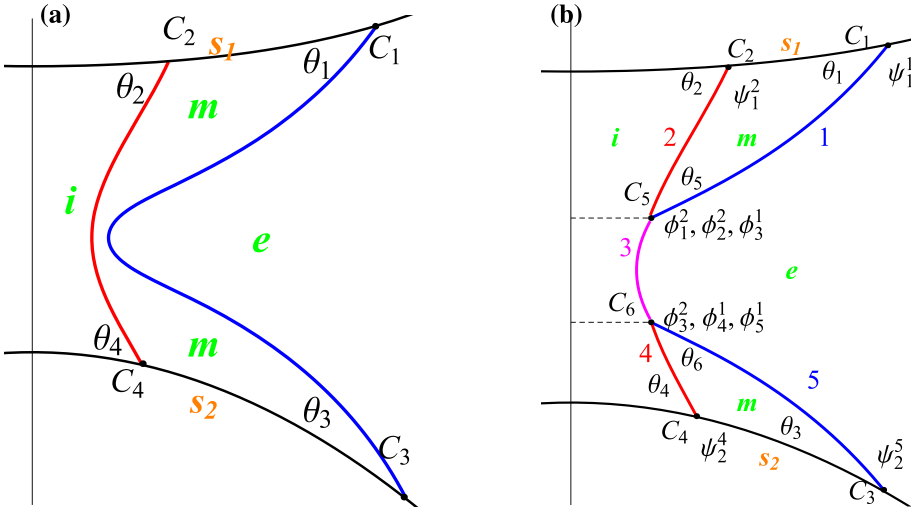

FIG. 1. (Color online) **a** The meridional section of two Und interfaces before LB$_m$ rupture. **b** Five Und interfaces of different curvatures for three immiscible liquids between two smooth solid bodies with free BC. The endpoints C_1, C_2, C_3, C_4 have one degree of freedom: the upper and lower endpoints are running along S_1 and S_2, respectively. The endpoints C_5, C_6 have two degrees of freedom and are located on two singular curves L_1, L_2, respectively, which are passing transversely to the plane of figure

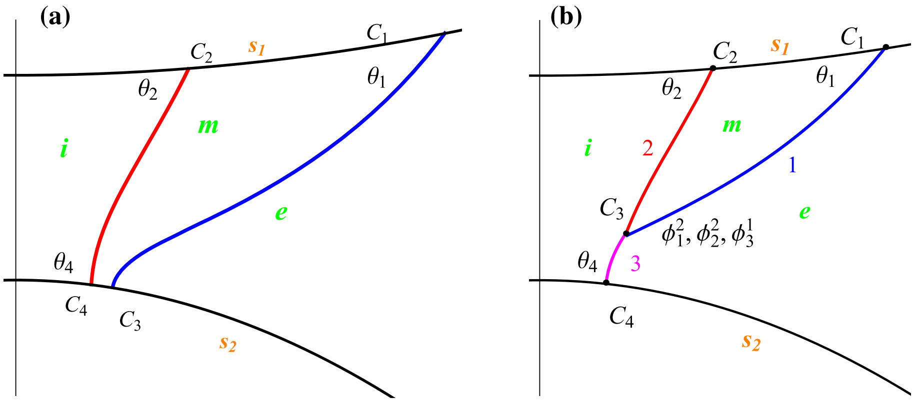

FIG. 2. (Color online) **a** Two Und interfaces before LB$_m$ rupture. **b** Three Und interfaces of different curvatures for three immiscible liquids trapped between two smooth solid bodies with free BCs. The endpoints C_1, C_2, C_4 have one degree of freedom while C_3 has two degrees and is located on a singular curve L which is passing transversely to the plane of figure

$$E[r,z] = \sum_{j=1}^{5} \int_{\phi_j^2}^{\phi_j^1} \mathsf{E}_j \mathrm{d}\phi_j + \int_{0}^{\psi_1^2} \mathsf{A}_{s_1}^{i} \mathrm{d}\psi_1 + \int_{\psi_1^2}^{\psi_1^1} \mathsf{A}_{s_1}^{m} \mathrm{d}\psi_1 + \int_{\psi_1^1}^{\infty} \mathsf{A}_{s_1}^{e} \mathrm{d}\psi_1$$

$$+ \int_{0}^{\psi_2^4} \mathsf{A}_{s_2}^{i} \psi_2 + \int_{\psi_2^4}^{\psi_2^5} \mathsf{A}_{s_2}^{m} \mathrm{d}\psi_2 + \int_{\psi_2^5}^{\infty} \mathsf{A}_{s_2}^{e} \psi_2, \tag{2.1}$$

$$\mathsf{E}_j = \gamma_j r_j \sqrt{r_j'^2 + z_j'^2}, \quad 1 \le j \le 5, \quad \mathsf{A}_{s_\alpha}^{l} = \gamma_{s_\alpha}^{l} R_\alpha \sqrt{R_\alpha'^2 + Z_\alpha'^2}, \tag{2.2}$$

where $r'_j = \mathrm{d}r_j/\mathrm{d}\phi_j$, $z'_j = \mathrm{d}z_j/\mathrm{d}\phi_j$, $R'_\alpha = \mathrm{d}R_\alpha/\mathrm{d}\psi_\alpha$ and $Z'_\alpha = \mathrm{d}Z_\alpha/\mathrm{d}\psi_\alpha$ and $\alpha = 1, 2$, $l = i, m, e$. Throughout the paper, the Latin and Greek indices enumerate the interfaces and solid surfaces, respectively. The surface tension coefficients $\gamma_1 = \gamma_5$, $\gamma_2 = \gamma_4$ and γ_3 denote tension at the e–m, m–i and e–i liquid interfaces, respectively, while $\gamma^l_{s_\alpha}$ stand for surface tension coefficients at the solid–liquid, s_α–l, interfaces (see Fig. 1b).

Two other functionals $V_i[r, z]$ and $V_m[r, z]$ for volumes of liquids i and m read

$$V_m[r,z] = \int_{\phi_1^2}^{\phi_1^1} \mathsf{V}_1 \mathrm{d}\phi_1 - \int_{\phi_2^2}^{\phi_2^1} \mathsf{V}_2 \mathrm{d}\phi_2 + \int_{\phi_5^2}^{\phi_5^1} \mathsf{V}_5 \mathrm{d}\phi_5 - \int_{\phi_4^2}^{\phi_4^1} \mathsf{V}_4 \mathrm{d}\phi_4 - \int_{\psi_1^2}^{\psi_1^1} \mathsf{B}_{s_1} \mathrm{d}\psi_1 + \int_{\psi_2^4}^{\psi_2^5} \mathsf{B}_{s_2} \mathrm{d}\psi_2,$$

$$V_i[r,z] = \int_{\phi_2^2}^{\phi_2^1} \mathsf{V}_2 \mathrm{d}\phi_2 + \int_{\phi_3^2}^{\phi_3^1} \mathsf{V}_3 \mathrm{d}\phi_3 + \int_{\phi_4^2}^{\phi_4^1} \mathsf{V}_4 \mathrm{d}\phi_4 - \int_{0}^{\psi_1^2} \mathsf{B}_{s_1} \mathrm{d}\psi_1 + \int_{0}^{\psi_2^4} \mathsf{B}_{s_2} \mathrm{d}\psi_2, \tag{2.3}$$

where

$$\mathsf{V}_j = \frac{1}{2} z'_j r_j^2, \quad 1 \le j \le 5, \quad \mathsf{B}_{s_\alpha} = \frac{1}{2} Z'_\alpha R_\alpha^2, \quad \alpha = 1, 2.$$

The isoperimetric problem [4] requires to find a set of functions $\bar{r}_j(\phi_j), \bar{z}_j(\phi_j)$, providing a local minimum of $E[r, z]$ with two constraints $V_i[r, z] = V_i$ and $V_m[r, z] = V_m$ imposed on the volumes of liquids i and m. Consider a composite functional

$$W[r,z] = E[r,z] - \lambda_1 V_m[r,z] - \lambda_3 V_i[r,z], \tag{2.4}$$

with two Lagrange multipliers λ_j and represent it in the following form,

$$W[r,z] = \sum_{j=1}^{5} \int_{\phi_j^2}^{\phi_j^1} \mathsf{F}_j \mathrm{d}\phi_1 + \int_{0}^{\psi_1^2} \mathsf{G}_1^i \mathrm{d}\psi_1 + \int_{\psi_1^2}^{\psi_1^1} \mathsf{G}_1^m \mathrm{d}\psi_1 + \int_{\psi_1^1}^{\infty} \mathsf{G}_1^e \mathrm{d}\psi_1 - \int_{0}^{\psi_2^4} \mathsf{G}_2^i \mathrm{d}\psi_2 - \int_{\psi_2^4}^{\psi_2^5} \mathsf{G}_2^m \mathrm{d}\psi_2 - \int_{\psi_2^5}^{\infty} \mathsf{G}_2^e \mathrm{d}\psi_2, \tag{2.5}$$

where $\mathsf{F}_j = \mathsf{F}(r_j, z_j, r'_j, z'_j)$ and $\mathsf{G}^l_\alpha(R_\alpha, Z_\alpha, R'_\alpha, Z'_\alpha)$ are given as follows,

$$\begin{aligned}
&\mathsf{F}_1 = \mathsf{E}_1 - \lambda_1 \mathsf{V}_1, \quad \mathsf{F}_2 = \mathsf{E}_2 - \lambda_2 \mathsf{V}_2, \qquad \mathsf{F}_3 = \mathsf{E}_3 - \lambda_3 \mathsf{V}_3,\\
&\mathsf{F}_4 = \mathsf{E}_4 - \lambda_4 \mathsf{V}_4, \quad \mathsf{F}_5 - \lambda_5 \mathsf{V}_5,\\
&\mathsf{G}_1^i = \lambda_3 \mathsf{B}_{s_1} + \mathsf{A}^i_{s_1}, \quad \mathsf{G}_1^m = \lambda_1 \mathsf{B}_{s_1} + \mathsf{A}^m_{s_1},\\
&\mathsf{G}_2^i = \lambda_3 \mathsf{B}_{s_2} - \mathsf{A}^i_{s_2}, \quad \mathsf{G}_2^m = \lambda_1 \mathsf{B}_{s_2} - \mathsf{A}^m_{s_2}, \quad \mathsf{G}_2^e = -\mathsf{A}^e_{s_2}, \quad \mathsf{G}_1^e = \mathsf{A}^e_{s_1},\\
&\lambda_2 = \lambda_3 - \lambda_1, \quad \lambda_5 = \lambda_1, \quad \lambda_4 = \lambda_2.
\end{aligned} \tag{2.6}$$

Calculate first variation of W when the functions $\bar{r}_j(\phi_j)$ and $\bar{z}_j(\phi_j)$ are perturbed by $u_j(\phi_j)$ and $v_j(\phi_j)$, respectively,

$$\delta W=\sum_{j=1}^{5}\int_{\phi_j^2}^{\phi_j^1}\Delta\mathsf{F}_j\,\mathrm{d}\phi_j+\left(\mathsf{G}_1^i-\mathsf{G}_1^m\right)\delta\psi_1^2+\left(\mathsf{G}_1^m-\mathsf{G}_1^e\right)\delta\psi_1^1 \tag{2.7}$$
$$-\left(\mathsf{G}_2^i-\mathsf{G}_2^m\right)\delta\psi_2^4-\left(\mathsf{G}_2^m-\mathsf{G}_2^e\right)\delta\psi_2^5,$$
$$\mathsf{G}_1^i-\mathsf{G}_1^m=\mathsf{A}_{s_1}^i-\mathsf{A}_{s_1}^m+\lambda_2\mathsf{B}_{s_1},\quad \mathsf{G}_1^m-\mathsf{G}_1^e=\mathsf{A}_{s_1}^m-\mathsf{A}_{s_1}^e+\lambda_1\mathsf{B}_{s_1},$$
$$\mathsf{G}_2^i-\mathsf{G}_2^m=\mathsf{A}_{s_2}^i-\mathsf{A}_{s_2}^m+\lambda_2\mathsf{B}_{s_2},\quad \mathsf{G}_2^m-\mathsf{G}_2^e=\mathsf{A}_{s_2}^m-\mathsf{A}_{s_21}^e+\lambda_1\mathsf{B}_{s_2},$$
$$\Delta\mathsf{F}_j=\frac{\partial\mathsf{F}_j}{\partial r_j}u_j+\frac{\partial\mathsf{F}_j}{\partial r_j'}u_j'+\frac{\partial\mathsf{F}_j}{\partial z_j}v_j+\frac{\partial\mathsf{F}_j}{\partial z_j'}v_j'. \tag{2.8}$$

The functions $u_j(\phi_j)$ and $v_j(\phi_j)$ may be derived using a requirement that the upper free endpoints of the first and second interfaces at Fig. 1b are running along S_1 and the lower free endpoints of the forth and fifth interfaces—along S_2 ,

$$\bar{r}_j(\phi_j^\alpha)=R_\alpha(\psi_\alpha^j),\quad \bar{r}_j(\phi_j^\alpha)+u_j(\phi_j^\alpha)=R_\alpha(\psi_\alpha^j+\delta\psi_\alpha^j), \tag{2.9}$$
$$u_j(\phi_j^\alpha)=\frac{\mathrm{d}R_\alpha}{\mathrm{d}\psi_\alpha}\delta\psi_\alpha^j,$$
$$\bar{z}_j(\phi_j^\alpha)=Z_\alpha(\psi_\alpha^j),\quad \bar{z}_j(\phi_j^\alpha)+v_j(\phi_j^\alpha)=Z_\alpha(\psi_\alpha^j+\delta\psi_\alpha^j), \tag{2.10}$$
$$v_j(\phi_j^\alpha)=\frac{\mathrm{d}Z_\alpha}{\mathrm{d}\psi_\alpha}\delta\psi_\alpha^j,$$

where $\alpha=1$ for $j=1,2$ and $\alpha=2$ for $j=4,5$. Substitute (2.8) into (2.7) and integrate by parts

$$\delta W=\sum_{j=1}^{5}\left[\int_{\phi_j^2}^{\phi_j^1}\left(u_j\frac{\delta\mathsf{F}_j}{\delta r_j}+v_j\frac{\delta\mathsf{F}_j}{\delta z_j}\right)\mathrm{d}\phi_j+\left(u_j\frac{\partial\mathsf{F}_j}{\partial r_j'}+v_j\frac{\partial\mathsf{F}_j}{\partial z_j'}\right)_{\phi_j^2}^{\phi_j^1}\right]$$
$$+\left(\mathsf{G}_1^i-\mathsf{G}_1^m\right)\delta\psi_1^2+\left(\mathsf{G}_1^m-\mathsf{G}_1^e\right)\delta\psi_1^1$$
$$-\left(\mathsf{G}_2^i-\mathsf{G}_2^m\right)\delta\psi_2^4-\left(\mathsf{G}_2^m-\mathsf{G}_2^e\right)\delta\psi_2^5, \tag{2.11}$$

where $\delta\mathsf{F}/\delta y_j=\partial\mathsf{F}/\partial y_j-\mathrm{d}/\mathrm{d}x\left(\partial\mathsf{F}/\partial y_j'\right)$, denotes the variational derivative for the functional $\int\mathsf{F}\left(x,y_j,y_j'\right)dx$. Since $u_j(\phi)$ and $v_j(\phi)$ are independent functions, vanishing of the integral part of δW in (2.9) gives rise to the Young–Laplace equations (YLE) [1],

$$\frac{\delta\mathsf{F}_j}{\delta r_j}=0,\quad \frac{\delta\mathsf{F}_j}{\delta z_j}=0\quad\rightarrow\quad \frac{z_j'}{r_j}+z_j''r_j'-z_j'r_j''=\frac{\lambda_j}{\gamma_j},\quad 1\le j\le 5. \tag{2.12}$$

Setting the remaining terms (2.11) to zero gives rise to the four transversality relations,

$$\frac{\mathrm{d}R_1}{\mathrm{d}\psi_1}\left(\psi_1^1\right)\frac{\partial\mathsf{F}_1}{\partial r_1'}\left(\phi_1^1\right)+\frac{\mathrm{d}Z_1}{\mathrm{d}\psi_1}\left(\psi_1^1\right)\frac{\partial\mathsf{F}_1}{\partial z_1'}\left(\phi_1^1\right)+\mathsf{G}_1^m\left(\psi_1^1\right)-\mathsf{G}_1^e\left(\psi_1^1\right)=0,$$
$$\frac{\mathrm{d}R_1}{\mathrm{d}\psi_1}\left(\psi_1^2\right)\frac{\partial\mathsf{F}_2}{\partial r_2'}\left(\phi_2^1\right)+\frac{\mathrm{d}Z_1}{\mathrm{d}\psi_1}\left(\psi_1^2\right)\frac{\partial\mathsf{F}_2}{\partial z_2'}\left(\phi_2^1\right)+\mathsf{G}_1^i\left(\psi_1^2\right)-\mathsf{G}_1^m\left(\psi_1^2\right)=0,$$
$$\frac{\mathrm{d}R_2}{\mathrm{d}\psi_2}\left(\psi_2^4\right)\frac{\partial\mathsf{F}_4}{\partial r_4'}\left(\phi_4^2\right)+\frac{\mathrm{d}Z_2}{\mathrm{d}\psi_2}\left(\psi_2^4\right)\frac{\partial\mathsf{F}_4}{\partial z_4'}\left(\phi_4^2\right)+\mathsf{G}_2^i\left(\psi_2^4\right)-\mathsf{G}_2^m\left(\psi_2^4\right)=0,$$
$$\frac{\mathrm{d}R_2}{\mathrm{d}\psi_2}\left(\psi_2^5\right)\frac{\partial\mathsf{F}_5}{\partial r_5'}\left(\phi_5^2\right)+\frac{\mathrm{d}Z_2}{\mathrm{d}\psi_2}\left(\psi_2^5\right)\frac{\partial\mathsf{F}_5}{\partial z_5'}\left(\phi_5^2\right)+\mathsf{G}_2^m\left(\psi_2^5\right)-\mathsf{G}_2^e\left(\psi_2^5\right)=0, \tag{2.13}$$

and one more transversality relation

$$\begin{aligned} & u_1\left(\phi_1^2\right)\frac{\partial \mathsf{F}_1}{\partial r_1'}+v_1\left(\phi_1^2\right)\frac{\partial \mathsf{F}_1}{\partial z_1'}+u_2\left(\phi_2^2\right)\frac{\partial \mathsf{F}_2}{\partial r_2'} \\ & \quad +v_2\left(\phi_2^2\right)\frac{\partial \mathsf{F}_2}{\partial z_2'}-u_3\left(\phi_3^1\right)\frac{\partial \mathsf{F}_3}{\partial r_3'}-v_3\left(\phi_3^1\right)\frac{\partial \mathsf{F}_3}{\partial z_3'} \\ & \quad +u_3\left(\phi_3^2\right)\frac{\partial \mathsf{F}_3}{\partial r_3'}+v_3\left(\phi_3^2\right)\frac{\partial \mathsf{F}_3}{\partial z_3'}-u_4\left(\phi_4^1\right)\frac{\partial \mathsf{F}_4}{\partial r_4'} \\ & \quad -v_4\left(\phi_4^1\right)\frac{\partial \mathsf{F}_4}{\partial z_4'}-u_5\left(\phi_5^1\right)\frac{\partial \mathsf{F}_5}{\partial r_5'}-v_5\left(\phi_5^1\right)\frac{\partial \mathsf{F}_5}{\partial z_5'}=0. \end{aligned} \tag{2.14}$$

In the case of one liquid bridge LB_m and two immiscible liquids m and e between two smooth solids S_1, S_2, the first and forth relations in (2.13) coincide with those derived in [1], formula (2.15), while the rest of relations disappear. Regarding condition (2.14), the perturbations $u_j\left(\phi_j^k\right)$ and $v_j\left(\phi_j^k\right)$ are related in such a way that the three disturbed interfaces $1,2,3$ (and other three $3,4,5$) always intersect at one point,

$$\begin{aligned} u_1\left(\phi_1^2\right)=u_2\left(\phi_2^2\right)=u_3\left(\phi_3^1\right), \quad v_1\left(\phi_1^2\right)=v_2\left(\phi_2^2\right)=v_3\left(\phi_3^1\right), \\ u_3\left(\phi_3^2\right)=u_4\left(\phi_4^1\right)=u_5\left(\phi_5^1\right), \quad v_3\left(\phi_3^2\right)=v_4\left(\phi_4^1\right)=v_5\left(\phi_5^1\right). \end{aligned} \tag{2.15}$$

Combine (2.14), (2.15) and use independence of $u_1\left(\phi_1^2\right), v_1\left(\phi_1^2\right), u_3\left(\phi_3^2\right), v_3\left(\phi_3^2\right)$ and obtain four relations,

$$\begin{aligned} \frac{\partial \mathsf{F}_1}{\partial r_1'}\left(\phi_1^2\right)+\frac{\partial \mathsf{F}_2}{\partial r_2'}\left(\phi_2^2\right)-\frac{\partial \mathsf{F}_3}{\partial r_3'}\left(\phi_3^1\right)=0, \\ \frac{\partial \mathsf{F}_1}{\partial z_1'}\left(\phi_1^2\right)+\frac{\partial \mathsf{F}_2}{\partial z_2'}\left(\phi_2^2\right)-\frac{\partial \mathsf{F}_3}{\partial z_3'}\left(\phi_3^1\right)=0, \\ \frac{\partial \mathsf{F}_3}{\partial r_3'}\left(\phi_3^2\right)-\frac{\partial \mathsf{F}_4}{\partial r_4'}\left(\phi_4^1\right)-\frac{\partial \mathsf{F}_5}{\partial r_5'}\left(\phi_5^1\right)=0, \\ \frac{\partial \mathsf{F}_3}{\partial z_3'}\left(\phi_3^2\right)-\frac{\partial \mathsf{F}_4}{\partial z_4'}\left(\phi_4^1\right)-\frac{\partial \mathsf{F}_5}{\partial z_5'}\left(\phi_5^1\right)=0. \end{aligned} \tag{2.16}$$

Boundary conditions (BC) (2.13, 2.15) have to be supplemented by condition of coincidence of interfaces in C_5, C_6 located on singular curves L_1, L_2, respectively,

$$\begin{aligned} r_1\left(\phi_1^2\right)=r_2\left(\phi_2^2\right)=r_3\left(\phi_3^1\right), \quad r_4\left(\phi_4^1\right)=r_5\left(\phi_5^1\right)=r_3\left(\phi_3^2\right), \\ z_1\left(\phi_1^2\right)=z_2\left(\phi_2^2\right)=z_3\left(\phi_3^1\right), \quad z_4\left(\phi_4^1\right)=z_5\left(\phi_5^1\right)=z_3\left(\phi_3^2\right), \end{aligned} \tag{2.17}$$

while the angular coordinates ϕ_j^k and ψ_α^j are related by

$$\begin{aligned} z_1\left(\phi_1^1\right)=Z_1\left(\psi_1^1\right), \quad r_1\left(\phi_1^1\right)=R_1\left(\psi_1^1\right), \\ z_2\left(\phi_2^1\right)=Z_1\left(\psi_1^2\right), \quad r_2\left(\phi_2^1\right)=R_1\left(\psi_1^2\right), \\ z_4\left(\phi_4^2\right)=Z_2\left(\psi_2^4\right), \quad r_4\left(\phi_4^2\right)=R_2\left(\psi_2^4\right), \\ z_5\left(\phi_5^2\right)=Z_2\left(\psi_2^5\right), \quad r_5\left(\phi_5^2\right)=R_2\left(\psi_2^5\right). \end{aligned} \tag{2.18}$$

Thus, we have 24 BC for the ten YLE (2.12) of the second order. Let us arrange them as follows,

$$\frac{\delta \mathsf{F}_1}{\delta r_1}=\frac{\delta \mathsf{F}_1}{\delta z_1}=0, \quad \begin{cases} \frac{\mathrm{d}R_1}{\mathrm{d}\psi_1}\left(\psi_1^1\right)\frac{\partial \mathsf{F}_1}{\partial r_1'}\left(\phi_1^1\right)+\frac{\mathrm{d}Z_1}{\mathrm{d}\psi_1}\left(\psi_1^1\right)\frac{\partial \mathsf{F}_1}{\partial z_1'}\left(\phi_1^1\right)+ \\ \qquad G_1^{me}\left(\psi_1^1\right)=0, \\ r_1\left(\phi_1^2\right)=r_3\left(\phi_3^1\right), \quad z_1\left(\phi_1^2\right)=z_3\left(\phi_3^1\right), \\ z_1\left(\phi_1^1\right)=Z_1\left(\psi_1^1\right), \quad r_1\left(\phi_1^1\right)=R_1\left(\psi_1^1\right), \end{cases} \tag{2.19}$$

$$\frac{\delta \mathsf{F}_2}{\delta r_2}=\frac{\delta \mathsf{F}_2}{\delta z_2}=0, \quad \begin{cases} \frac{\mathrm{d}R_1}{\mathrm{d}\psi_1}\left(\psi_1^2\right)\frac{\partial \mathsf{F}_2}{\partial r_2'}\left(\phi_2^1\right)+\frac{\mathrm{d}Z_1}{\mathrm{d}\psi_1}\left(\psi_1^2\right)\frac{\partial \mathsf{F}_2}{\partial z_2'}\left(\phi_2^1\right)+ \\ \qquad G_1^{im}\left(\psi_1^2\right)=0, \\ r_2\left(\phi_2^2\right)=r_3\left(\phi_3^1\right), \quad z_2\left(\phi_2^2\right)=z_3\left(\phi_3^1\right), \\ z_2\left(\phi_2^1\right)=Z_1\left(\psi_1^2\right), \quad r_2\left(\phi_2^1\right)=R_1\left(\psi_1^2\right), \end{cases} \tag{2.20}$$

$$\frac{\delta \mathsf{F}_3}{\delta r_3}=\frac{\delta \mathsf{F}_3}{\delta z_3}=0, \quad \begin{cases} \frac{\partial \mathsf{F}_1}{\partial r_1'}\left(\phi_1^2\right)+\frac{\partial \mathsf{F}_2}{\partial r_2'}\left(\phi_2^2\right)-\frac{\partial \mathsf{F}_3}{\partial r_3'}\left(\phi_3^1\right)=0, \\ \frac{\partial \mathsf{F}_1}{\partial z_1'}\left(\phi_1^2\right)+\frac{\partial \mathsf{F}_2}{\partial z_2'}\left(\phi_2^2\right)-\frac{\partial \mathsf{F}_3}{\partial z_3'}\left(\phi_3^1\right)=0, \\ \frac{\partial \mathsf{F}_3}{\partial r_3'}\left(\phi_3^2\right)-\frac{\partial \mathsf{F}_4}{\partial r_4'}\left(\phi_4^1\right)-\frac{\partial \mathsf{F}_5}{\partial r_5'}\left(\phi_5^1\right)=0, \\ \frac{\partial \mathsf{F}_3}{\partial z_3'}\left(\phi_3^2\right)-\frac{\partial \mathsf{F}_4}{\partial z_4'}\left(\phi_4^1\right)-\frac{\partial \mathsf{F}_5}{\partial z_5'}\left(\phi_5^1\right)=0, \end{cases} \tag{2.21}$$

$$\frac{\delta \mathsf{F}_4}{\delta r_4}=\frac{\delta \mathsf{F}_4}{\delta z_4}=0, \quad \begin{cases} \frac{\mathrm{d}R_2}{\mathrm{d}\psi_2}\left(\psi_2^4\right)\frac{\partial \mathsf{F}_4}{\partial r_4'}\left(\phi_4^2\right)+\frac{\mathrm{d}Z_2}{\mathrm{d}\psi_2}\left(\psi_2^4\right)\frac{\partial \mathsf{F}_4}{\partial z_4'}\left(\phi_4^2\right)+ \\ \qquad G_2^{im}\left(\psi_2^4\right)=0, \\ r_4\left(\phi_4^1\right)=r_3\left(\phi_3^2\right), \quad r_4\left(\phi_4^1\right)=r_3\left(\phi_3^2\right), \\ z_4\left(\phi_4^2\right)=Z_2\left(\psi_2^4\right), \quad r_4\left(\phi_4^2\right)=R_2\left(\psi_2^4\right), \end{cases} \tag{2.22}$$

$$\frac{\delta \mathsf{F}_5}{\delta r_5}=\frac{\delta \mathsf{F}_5}{\delta z_5}=0, \quad \begin{cases} \frac{\mathrm{d}R_2}{\mathrm{d}\psi_2}\left(\psi_2^5\right)\frac{\partial \mathsf{F}_5}{\partial r_5'}\left(\phi_5^2\right)+\frac{\mathrm{d}Z_2}{\mathrm{d}\psi_2}\left(\psi_2^5\right)\frac{\partial \mathsf{F}_5}{\partial z_5'}\left(\phi_5^2\right)+ \\ \qquad G_2^{me}\left(\psi_2^5\right)=0, \\ r_5\left(\phi_5^1\right)=r_3\left(\phi_3^2\right), \quad z_5\left(\phi_5^1\right)=z_3\left(\phi_3^2\right), \\ z_5\left(\phi_5^2\right)=Z_2\left(\psi_2^5\right), \quad r_5\left(\phi_5^2\right)=R_2\left(\psi_2^5\right), \end{cases} \tag{2.23}$$

where

$$\begin{aligned} G_1^{me}\left(\psi_1^1\right)&=\mathsf{G}_1^m\left(\psi_1^1\right)-\mathsf{G}_1^e\left(\psi_1^1\right), \qquad & G_1^{im}\left(\psi_1^2\right)&=\mathsf{G}_1^i\left(\psi_1^2\right)-\mathsf{G}_1^m\left(\psi_1^2\right), \\ G_2^{im}\left(\psi_2^4\right)&=\mathsf{G}_2^i\left(\psi_2^4\right)-\mathsf{G}_2^m\left(\psi_2^4\right), \qquad & G_2^{me}\left(\psi_2^5\right)&=\mathsf{G}_2^m\left(\psi_2^5\right)-\mathsf{G}_2^e\left(\psi_2^5\right). \end{aligned}$$

2.1. Curvature law and interface consistency rules

Analysis of YLE (2.12) yields an important conclusion about the curvatures H_j of five interfaces. Consider (2.12) and recall that according to [1], $\lambda_j = 2\gamma_j H_j$. Combining this scaling with (2.6), we arrive at relationships between the curvatures of three interfaces,

$$\gamma_1 H_1 + \gamma_2 H_2 = \gamma_3 H_3, \qquad H_1 = H_5, \quad H_2 = H_4. \tag{2.24}$$

Simple verification of (2.24) can be done in special cases. Indeed, if the liquids i and m are indistinguishable, i.e., $\gamma_1 = \gamma_3$ and $\gamma_2 = 0$, then $H_1 = H_3$. On the other hand, if the liquids m and e are indistinguishable, i.e., $\gamma_2 = \gamma_3$ and $\gamma_1 = 0$, then $H_2 = H_3$. In the case $\gamma_1 = \gamma_2 = \gamma_3 \neq 0$, we arrive at relation known in theory of double bubble [6] when three spherical soap surfaces meet at a contact line.

Interfaces	Mns$^+$	Cat	Nod$^-$
Mns$^+$	Mns$^+$	Mns$^+$	Mns$^\pm$
Cat	Mns$^+$	Cat	Nod$^-$
Nod$^-$	Mns$^\pm$	Nod$^-$	Nod$^-$

We can formulate strong statements on consistency of five interfaces based on relations (2.24). Recall [1] that there exists only one type, Nod$^-$, of interfaces with negative H, while the other interfaces have positive H: nodoid Nod$^+$, cylinder Cyl, unduloid Und, sphere Sph, or zero curvature, catenoid Cat. Denote by $\mathsf{Mns}^+ = \{\mathsf{Nod}^+, \mathsf{Cyl}, \mathsf{Und}, \mathsf{Sph}\}$ a set of interfaces with $H > 0$ and by $\mathsf{Mns}^\pm = \{\mathsf{Mns}^+, \mathsf{Cat}, \mathsf{Nod}^-\}$ a set of all admissible interfaces. The rules of interfaces consistency with curvatures H_1, H_2, H_3 are given in Table, e.g., if the first and second interfaces are Cat and Nod$^-$ then the third interface has to be also Nod$^-$, but if the first and second interfaces are Und and Nod$^-$ then the third interface may be any of six interfaces.

2.2. Standard parameterization and symmetric setup

Consider nonzero curvature interfaces $r_j(\phi_j)$, $z_j(\phi_j)$, $1 \le j \le 5$, between two solid bodies, $\{R_\alpha(\psi_\alpha), Z_\alpha(\psi_\alpha)\}$, $\alpha = 1, 2$, and choose interfaces parameterization in such a way that the lower ϕ_j^2 and the upper ϕ_j^1 coordinates of endpoints C_1, C_2, C_3, C_4 are located on the solid surfaces S_1, S_2 and governed by BC while the other two points C_5, C_6 denote the triple points located on singular curves L_1, L_2 where three different interfaces meet together.

Following [1] write the parametric expressions for the shape of such interfaces $z_j(\phi_j)$ and $r_j(\phi_j)$,

$$z_j(\phi_j) = \frac{M(\phi_j, B_j)}{2|H_j|} + d_j, \quad r_j(\phi_j) = \frac{1}{2|H_j|}\sqrt{1 + B_j^2 + 2B_j \cos\phi_j}, \tag{2.25}$$

$$M(\phi, B) = (1+B)E\left(\frac{\phi}{2}, m\right) + (1-B)F\left(\frac{\phi}{2}, m\right), \quad m^2 = \frac{4B}{(1+B)^2},$$

where

$$r_j' = -\frac{B_j \sin\phi_j}{2|H_j|r_j}, \quad z_j' = \frac{1 + B_j\cos\phi_j}{2|H_j|r_j}, \quad \frac{z_j'}{r_j'} = -\frac{1 + B_j\cos\phi_j}{B_j \sin\phi_j}, \tag{2.26}$$

and $r_j'^2 + z_j'^2 = 1$. For all interfaces, we have to find 24 unknowns: $15 - 1 = 14$ interfaces parameters H_j, B_j, d_j [due to (2.24)] and 10 endpoint values ϕ_j^1, ϕ_j^2. These unknowns have to satisfy 24 BCs in (2.19–2.23).

When both surfaces S_1 and S_2 are similar and the picture in Fig. 1b is symmetric w.r.t. midline between S_1 and S_2, then such setup reduces the problem above to 6 YLE (2.19–2.21) for the first, second and third interfaces with 12 unknowns:

$$\phi_1^1,\ \phi_2^1,\ \phi_3^1,\ \phi_1^2,\ \phi_2^2,\ d_1,\ d_2,\ B_1,\ B_2,\ B_3,\ H_1,\ H_2,$$

and

$$\phi_3^2 = \pi, \quad 2d_3 = -M(\pi, B_3)/|H_3|, \quad H_3 = (\gamma_1 H_1 + \gamma_2 H_2)/\gamma_3.$$

This number coincides with 12 BCs which comprise 10 BCs in (2.20, 2.21) and two first BCs in (2.22). Calculate the partial derivatives $\partial \mathsf{F}_j/\partial r_j'$, $\partial \mathsf{F}_j/\partial z_j'$ and write these twelve BCs,

$$\begin{aligned}
&\gamma_1 r_1 \left(r_1' R_1' + z_1' Z_1'\right) + \left(\gamma_{s_1}^m - \gamma_{s_1}^e\right) R_1 \sqrt{R_1'^2 + Z_1'^2} + \lambda_1 Z_1' \frac{R_1^2 - r_1^2}{2} = 0,\\
&\phi_1 = \phi_1^1, \quad \psi_1 = \psi_1^1,\\
&r_1\left(\phi_1^2\right) = r_3\left(\phi_3^1\right), \quad z_1\left(\phi_1^2\right) = z_3\left(\phi_3^1\right),\\
&z_1\left(\phi_1^1\right) = Z_1\left(\psi_1^1\right), \quad r_1\left(\phi_1^1\right) = R_1\left(\psi_1^1\right),
\end{aligned} \tag{2.27}$$

$$\gamma_2 r_2 \left(r_2' R_1' + z_2' Z_1'\right) + \left(\gamma_{s_1}^i - \gamma_{s_1}^m\right) R_1 \sqrt{R_1'^2 + Z_1'^2} + \lambda_2 Z_1' \frac{R_1^2 - r_2^2}{2} = 0,$$
$$\phi_2 = \phi_2^1, \quad \psi_1 = \psi_1^2,$$
$$r_2\left(\phi_2^2\right) = r_3\left(\phi_3^1\right), \quad z_2\left(\phi_2^2\right) = z_3\left(\phi_3^1\right),$$
$$z_2\left(\phi_2^1\right) = Z_1\left(\psi_1^2\right), \quad r_2\left(\phi_2^1\right) = R_1\left(\psi_1^2\right), \tag{2.28}$$
$$\gamma_1 r_1 r_1' + \gamma_2 r_2 r_2' - \gamma_3 r_3 r_3' = 0, \quad \phi_1 = \phi_1^2,\ \phi_2 = \phi_2^2,\ \phi_3 = \phi_3^1,$$
$$\gamma_1 r_1 z_1' + \gamma_2 r_2 z_2' - \gamma_3 r_3 z_3' = \frac{1}{2}\left(\lambda_1 r_1^2 + \lambda_2 r_2^2 - \lambda_3 r_3^2\right). \tag{2.29}$$

After simplification we obtain

$$\gamma_1 \cos\theta_1^1 + \gamma_{s_1}^m - \gamma_{s_1}^e = 0, \quad \gamma_2 \cos\theta_1^2 + \gamma_{s_1}^i - \gamma_{s_1}^m = 0, \tag{2.30}$$
$$r_1\left(\phi_1^2\right) = r_2\left(\phi_2^2\right) = r_3\left(\phi_3^1\right),$$
$$z_1\left(\phi_1^1\right) = Z_1\left(\psi_1^1\right), \quad r_1\left(\phi_1^1\right) = R_1\left(\psi_1^1\right),$$
$$z_1\left(\phi_1^2\right) = z_2\left(\phi_2^2\right) = z_3\left(\phi_3^1\right),$$
$$z_2\left(\phi_2^1\right) = Z_1\left(\psi_1^2\right), \quad r_2\left(\phi_2^1\right) = R_1\left(\psi_1^2\right),$$
$$\gamma_1 r_1'\left(\phi_1^2\right) + \gamma_2 r_2'\left(\phi_2^2\right) - \gamma_3 r_3'\left(\phi_3^1\right) = 0,$$
$$\gamma_1 z_1'\left(\phi_1^2\right) + \gamma_2 z_2'\left(\phi_2^2\right) - \gamma_3 z_3'\left(\phi_3^1\right) = 0, \tag{2.31}$$

where $\cos\theta_1^j = \left(r_j' R_1' + z_j' Z_1'\right)/\sqrt{R_1'^2 + Z_1'^2}$ determines the contact angle θ_1^j of the jth interface and S_1. Two equalities in (2.30) give the Young relations at the points C_1, C_2 on S_1, while two equalities in (2.31) represent the Neumann triangle relations at the triple point C_5 located on a singular curve [8]. Indeed, the latter equalities are the r and z projections of the vectorial equality for capillary forces $\mathbf{f}_j$ at C_5 in outward directions w.r.t. C_5 and tangential to meridional section of menisci,

$$\mathbf{f}_1(C_5) + \mathbf{f}_2(C_5) + \mathbf{f}_3(C_5) = 0, \quad \mathbf{f}_j(C_5) = \gamma_j \left\{r_j'(C_5), z_j'(C_5)\right\}. \tag{2.32}$$

Finish this section with one more observation related to the surface tensions γ_j and contact angles of three interfaces on solid surface. Bearing in mind that $\gamma_3 \cos\theta_1^3 + \gamma_{s_1}^i - \gamma_{s_1}^e = 0$, combine the last equality with two others in (2.30) and obtain,

$$\gamma_1 \cos\theta_1^1 + \gamma_2 \cos\theta_1^2 = \gamma_3 \cos\theta_1^3. \tag{2.33}$$

2.3. Solving the BC equations (liquid bridges between two parallel plates)

Making use of standard parametrization (2.25), we present below 12 BCs (2.30, 2.31) for 12 unknowns $\phi_1^1, \phi_2^1, \phi_3^1, \phi_1^2, \phi_2^2, d_1, d_2, B_1, B_2, B_3, H_1, H_2$, in a way convenient for numerical calculations,

$$\frac{\gamma_1 B_1 \sin\phi_1^2}{|H_1|} + \frac{\gamma_2 B_2 \sin\phi_2^2}{|H_2|} = \frac{\gamma_3 B_3 \sin\phi_3^1}{|H_3|},$$
$$\frac{\gamma_1 B_1 \cos\phi_1^2}{|H_1|} + \frac{\gamma_2 B_2 \cos\phi_2^2}{|H_2|} = \frac{\gamma_3 B_3 \cos\phi_3^1}{|H_3|},$$
$$\frac{\sqrt{1 + 2B_j \cos\phi_j^2 + B_j^2}}{|H_j|} = \frac{\sqrt{1 + 2B_3 \cos\phi_3^1 + B_3^2}}{|H_3|}, \quad j = 1, 2, \tag{2.34}$$
$$\frac{M(\phi_1^2, B_1)}{2|H_1|} + d_1 = \frac{M(\phi_2^2, B_2)}{2|H_2|} + d_2 = \frac{M(\phi_3^1, B_3)}{2|H_3|} + d_3,$$
$$d_3 = -\frac{M(\pi, B_3)}{2|H_3|},$$

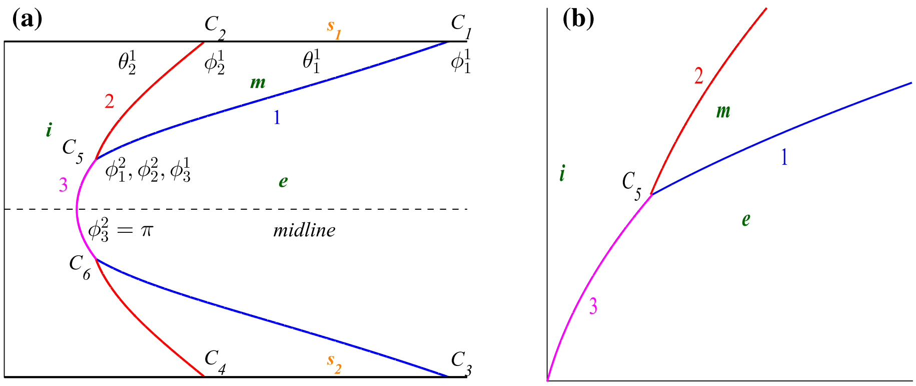

Fig. 3. (Color online) **a** Five Und interfaces for three immiscible media (i—water; m—octane; e—air) trapped between two similar solid plates ($Z_1 - Z_2 = 2$) with free BC. **b** Enlarged view of the vicinity of the triple point C_5 on the singular curve where three phases coexist. The angles between the adjacent interfaces read: $\Phi_{12} = 36.76°$, $\Phi_{23} = 159.43°$, $\Phi_{31} = 163.81°$

Interfaces	γ_j (mN/m)	θ_j	B_j	H_j	d_j	ϕ_j^2	ϕ_j^1
1 (e–m)	21.8	19°	0.959	0.095	−11.424	184.39°	208.74°
2 (m–i)	50.8	39°	0.778	0.272	−4.664	186.405°	216.34°
3 (e–i)	72.8	34.4°	0.855	0.218	−5.599	180°	188.14°

$$\frac{M(\phi_j^1, B_j) - M(\phi_j^2, B_j)}{2|H_j|} = Z_1\left(\psi_1^j\right) - \frac{M(\phi_3^1, B_3)}{2|H_3|} - d_3,$$

$$|H_j| = \frac{\sqrt{1 + 2B_j \cos\phi_j^1 + B_j^2}}{2R_1\left(\psi_1^j\right)},\ B_j = \left[\cos\phi_j^1 + \sin\phi_j^1 \tan\theta_1^j\right]^{-1},\ j = 1, 2,$$

where $H_3 = H_1\gamma_1/\gamma_3 + H_2\gamma_2/\gamma_3$.

The numerical optimization of the solution was done by a standard gradient descent algorithm. The cost function for the optimization problem was chosen to be the weighted sum of absolute values of the differences between the right- and the left-hand sides of the six first equations in (2.34).

In Fig. 3, we present the shapes of five interfaces of different curvatures for three immiscible media: i—water, m—octane (C_8H_{18}, a component of petrol), and e—air, trapped between two similar glass plates with free BCs and capillary parameters taken from [3]. The parameters of the interfaces are given in Table below Fig. 3. The volumes of liquids confined inside interfaces read $V_m = 4.009$, $V_i = 2.674$.

3. Variational problem for three interfaces

Consider a functional $E[r, z]$ of surface energy

$$E[r, z] = \sum_{j=1}^{3} \int_{\phi_j^2}^{\phi_j^1} \mathsf{E}_j \mathrm{d}\phi_j + \int_0^{\psi_1^2} \mathsf{A}_{s_1}^i \mathrm{d}\psi_1 + \int_{\psi_1^2}^{\psi_1^1} \mathsf{A}_{s_1}^m \mathrm{d}\psi_1 \tag{3.1}$$

$$+ \int_{\psi_1^1}^{\infty} \mathsf{A}_{s_1}^e \mathrm{d}\psi_1 + \int_0^{\psi_2^3} \mathsf{A}_{s_2}^i \psi_2 + \int_{\psi_2^3}^{\infty} \mathsf{A}_{s_2}^e \psi_2,$$

and two functionals $V_i[r,z]$ and $V_m[r,z]$ of volumes of the i and m liquids

$$V_m[r,z] = \int_{\phi_1^2}^{\phi_1^1} \mathsf{V}_1 \mathrm{d}\phi_1 - \int_{\phi_2^2}^{\phi_2^1} \mathsf{V}_2 \mathrm{d}\phi_2 - \int_{\psi_1^2}^{\psi_1^1} \mathsf{B}_{s_1} \mathrm{d}\psi_1,$$
$$V_i[r,z] = \int_{\phi_2^2}^{\phi_2^1} \mathsf{V}_2 \mathrm{d}\phi_2 + \int_{\phi_3^2}^{\phi_3^1} \mathsf{V}_3 \mathrm{d}\phi_3 - \int_0^{\psi_1^2} \mathsf{B}_{s_1} \mathrm{d}\psi_1 + \int_0^{\psi_2^3} \mathsf{B}_{s_2} \mathrm{d}\psi_2, \tag{3.2}$$

where all integrands E_j, $\mathsf{A}_{s_\alpha}^{i,m,e}$, V_j and B_{s_α} are defined in (2.2, 2.4).

Consider the composed functional $W[r,z] = E[r,z] - \lambda_1 V_m[r,z] - \lambda_3 V_i[r,z]$ and represent it in the following form,

$$W[r,z] = \sum_{j=1}^{3} \int_{\phi_j^2}^{\phi_j^1} \mathsf{F}_j \mathrm{d}\phi_1 + \int_0^{\psi_1^2} \mathsf{G}_1^i \mathrm{d}\psi_1 + \int_{\psi_1^2}^{\psi_1^1} \mathsf{G}_1^m \mathrm{d}\psi_1 + \int_{\psi_1^1}^{\infty} \mathsf{G}_1^e \mathrm{d}\psi_1$$
$$- \int_0^{\psi_2^3} \mathsf{G}_2^i \mathrm{d}\psi_2 - \int_{\psi_2^3}^{\infty} \mathsf{G}_2^e \mathrm{d}\psi_2, \tag{3.3}$$

where the integrands are given in (2.6).

Applying a similar technique as in Sect. 2, we arrive at the first variation,

$$\delta W = \sum_{j=1}^{5} \left[\int_{\phi_j^2}^{\phi_j^1} \left(u_j \frac{\delta \mathsf{F}_j}{\delta r_j} + v_j \frac{\delta \mathsf{F}_j}{\delta z_j} \right) \mathrm{d}\phi_j + \left(u_j \frac{\partial \mathsf{F}_j}{\partial r_j'} + v_j \frac{\partial \mathsf{F}_j}{\partial z_j'} \right)_{\phi_j^2}^{\phi_j^1} \right]$$
$$+ \left(\mathsf{G}_1^i - \mathsf{G}_1^m\right) \delta\psi_1^2 + \left(\mathsf{G}_1^m - \mathsf{G}_1^e\right) \delta\psi_1^1 - \left(\mathsf{G}_2^i - \mathsf{G}_2^e\right) \delta\psi_2^3. \tag{3.4}$$

This case does not allow a symmetric version and therefore is less reducible compared to the case of 5 interfaces w.r.t. the number of unknowns and BC equations. This number is equal 15: 9 interface parameters H_j, B_j, d_j, and 6 endpoint values ϕ_j^1, ϕ_j^2. They satisfy 15 BC equations

$$\frac{\delta \mathsf{F}_1}{\delta r_1} = \frac{\delta \mathsf{F}_1}{\delta z_1} = 0, \quad \begin{cases} \frac{dR_1}{\mathrm{d}\psi_1}\left(\psi_1^1\right) \frac{\partial \mathsf{F}_1}{\partial r_1'}\left(\phi_1^1\right) + \frac{\mathrm{d}Z_1}{\mathrm{d}\psi_1}\left(\psi_1^1\right) \frac{\partial \mathsf{F}_1}{\partial z_1'}\left(\phi_1^1\right) + \\ \qquad G_1^{me}\left(\psi_1^1\right) = 0, \\ r_1\left(\phi_1^2\right) = r_3\left(\phi_3^1\right), \quad z_1\left(\phi_1^2\right) = z_3\left(\phi_3^1\right), \\ z_1\left(\phi_1^1\right) = Z_1\left(\psi_1^1\right), \quad r_1\left(\phi_1^1\right) = R_1\left(\psi_1^1\right), \end{cases}$$
$$\frac{\delta \mathsf{F}_2}{\delta r_2} = \frac{\delta \mathsf{F}_2}{\delta z_2} = 0, \quad \begin{cases} \frac{dR_1}{\mathrm{d}\psi_1}\left(\psi_1^2\right) \frac{\partial \mathsf{F}_2}{\partial r_2'}\left(\phi_2^1\right) + \frac{\mathrm{d}Z_1}{\mathrm{d}\psi_1}\left(\psi_1^2\right) \frac{\partial \mathsf{F}_2}{\partial z_2'}\left(\phi_2^1\right) + \\ \qquad G_1^{im}\left(\psi_1^2\right) = 0, \\ r_2\left(\phi_2^2\right) = r_3\left(\phi_3^1\right), \quad z_2\left(\phi_2^2\right) = z_3\left(\phi_3^1\right), \\ z_2\left(\phi_2^1\right) = Z_1\left(\psi_1^2\right), \quad r_2\left(\phi_2^1\right) = R_1\left(\psi_1^2\right), \end{cases} \tag{3.5}$$

$$\frac{\delta \mathsf{F}_3}{\delta r_3} = \frac{\delta \mathsf{F}_3}{\delta z_3} = 0, \quad \begin{cases} \frac{\partial \mathsf{F}_1}{\partial r_1'}\left(\phi_1^2\right) + \frac{\partial \mathsf{F}_2}{\partial r_2'}\left(\phi_2^2\right) - \frac{\partial \mathsf{F}_3}{\partial r_3'}\left(\phi_3^1\right) = 0, \\ \frac{\partial \mathsf{F}_1}{\partial z_1'}\left(\phi_1^2\right) + \frac{\partial \mathsf{F}_2}{\partial z_2'}\left(\phi_2^2\right) - \frac{\partial \mathsf{F}_3}{\partial z_3'}\left(\phi_3^1\right) = 0, \\ \frac{dR_2}{d\psi_2}\left(\psi_2^3\right)\frac{\partial \mathsf{F}_5}{\partial r_5'}\left(\phi_3^2\right) + \frac{dZ_2}{d\psi_2}\left(\psi_2^3\right)\frac{\partial \mathsf{F}_5}{\partial z_5'}\left(\phi_3^2\right) + \\ \qquad G_2^{ie}\left(\psi_2^3\right) = 0, \\ z_5\left(\phi_5^2\right) = Z_2\left(\psi_2^3\right), \quad r_5\left(\phi_5^2\right) = R_2\left(\psi_2^3\right), \end{cases}$$

that gives

$$\begin{aligned} &\gamma_1 \cos\theta_1^1 + \gamma_{s_1}^m - \gamma_{s_1}^e = 0, \quad \gamma_2 \cos\theta_1^2 + \gamma_{s_1}^i - \gamma_{s_1}^m = 0, \\ &\gamma_3 \cos\theta_2^3 + \gamma_{s_2}^i - \gamma_{s_2}^e = 0, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\gamma_1 r_1'\left(\phi_1^2\right) + \gamma_2 r_2'\left(\phi_2^2\right) - \gamma_3 r_3'\left(\phi_3^1\right) = 0, \\ &\gamma_1 z_1'\left(\phi_1^2\right) + \gamma_2 z_2'\left(\phi_2^2\right) - \gamma_3 z_3'\left(\phi_3^1\right) = 0, \\ &r_1\left(\phi_1^2\right) = r_2\left(\phi_2^2\right) = r_3\left(\phi_3^1\right), \quad z_1\left(\phi_1^2\right) = z_2\left(\phi_2^2\right) = z_3\left(\phi_3^1\right), \\ &r_1\left(\phi_1^1\right) = R_1\left(\psi_1^1\right), \quad r_2\left(\phi_2^1\right) = R_1\left(\psi_1^2\right), \quad r_3\left(\phi_3^2\right) = R_2\left(\psi_2^3\right), \\ &z_1\left(\phi_1^1\right) = Z_1\left(\psi_1^1\right), \quad z_2\left(\phi_2^1\right) = Z_1\left(\psi_1^2\right), \quad z_3\left(\phi_3^2\right) = Z_2\left(\psi_2^3\right). \end{aligned} \tag{3.7}$$

Three equalities in (3.6) cannot be reduced to a single equality similar to (2.33) if the upper and lower solid bodies have different capillary properties, namely $\gamma_{s_2}^i - \gamma_{s_1}^i \neq \gamma_{s_2}^e - \gamma_{s_1}^e$, i.e.,

$$\gamma_1 \cos\theta_1^1 + \gamma_2 \cos\theta_1^2 \neq \gamma_3 \cos\theta_2^3.$$

3.1. Solving the BC equations (liquid bridges between two parallel plates)

Using a standard parametrization (2.25) and relation (2.24) for H_3, we present below 14 BCs (3.6,3.7) for 14 unknowns:

$$\phi_1^1,\ \phi_2^1,\ \phi_3^1,\ \phi_1^2,\ \phi_2^2,\ \phi_3^2,\ d_1,\ d_2,\ d_3,\ B_1,\ B_2,\ B_3,\ H_1,\ H_2,$$

in a way convenient for numerical calculations,

$$\begin{aligned} &B_j = [\cos\phi_j^1 + \sin\phi_j^1 \tan\theta_1^j]^{-1}, \ \frac{M(\phi_j^1, B_j)}{2|H_j|} + d_j = Z_1\left(\psi_1^j\right), \ j = 1, 2, \\ &B_3 = [\cos\phi_3^2 + \sin\phi_3^2 \tan\theta_2^3]^{-1}, \quad \frac{M(\phi_3^2, B_3)}{2|H_3|} + d_3 = Z_2\left(\psi_2^3\right), \\ &|H_1| = \frac{\sqrt{1 + 2B_1\cos\phi_1^1 + B_1^2}}{2R_1\left(\psi_1^1\right)}, \quad |H_2| = \frac{\sqrt{1 + 2B_2\cos\phi_2^1 + B_2^2}}{2R_1\left(\psi_1^2\right)}, \\ &\frac{\gamma_1 B_1 \sin\phi_1^2}{|H_1|} + \frac{\gamma_2 B_2 \sin\phi_2^2}{|H_2|} = \frac{\gamma_3 B_3 \sin\phi_3^1}{|H_3|}, \\ &\frac{\gamma_1 B_1 \cos\phi_1^2}{|H_1|} + \frac{\gamma_2 B_2 \cos\phi_2^2}{|H_2|} = \frac{\gamma_3 B_3 \cos\phi_3^1}{|H_3|}, \\ &\frac{\sqrt{1 + 2B_j\cos\phi_j^2 + B_j^2}}{|H_1|} = \frac{\sqrt{1 + 2B_3\cos\phi_3^1 + B_3^2}}{|H_3|}, \quad j = 1, 2, \\ &\frac{M(\phi_1^2, B_1)}{2|H_1|} + d_1 = \frac{M(\phi_2^2, B_2)}{2|H_2|} + d_2 = \frac{M(\phi_3^1, B_3)}{2|H_3|} + d_3, \end{aligned} \tag{3.8}$$

where $H_3 = (H_1\gamma_1 + H_2\gamma_2)/\gamma_3$. In Fig. 4, we present the shapes of three interfaces of different curvatures for three immiscible media: i—water, m—hexane (C_6H_{14}, a component of petrol), and e—air, trapped

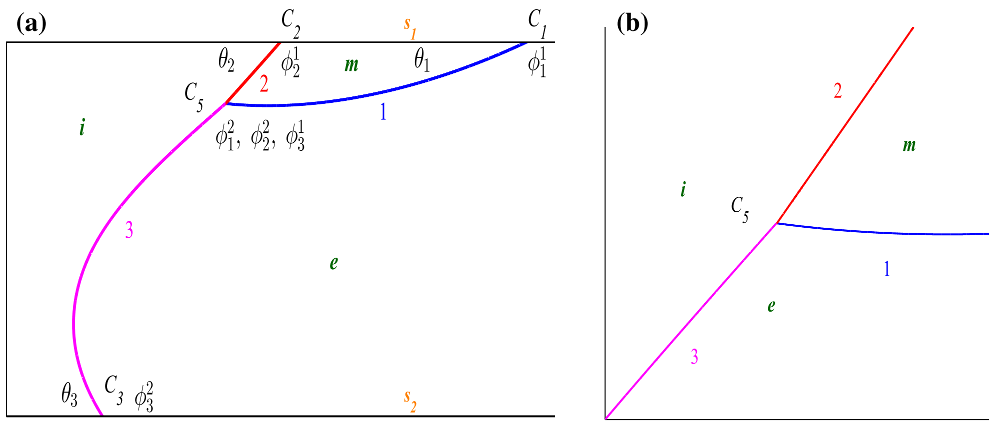

Fig. 4. (Color online) **a** Two Und and one Nod interfaces for three immiscible media (i—water, m—hexane, and e—air) trapped between two (not similar) solid plates ($Z_1 - Z_2 = 1$) with free BC. **b** Enlarged view of the vicinity of the triple point C_5 on singular curve L where three phases coexist. The angles between the adjacent interfaces reach the following values: $\Phi_{12} = 46.04°$, $\Phi_{23} = 173.28°$, $\Phi_{31} = 140.68°$

Interfaces	γ_j (mN/m)	θ_j	B_j	H_j	d_j	ϕ_j^2	ϕ_j^1
1 (e–m)	18.4	19°	1.091	0.229	−2.521	199.51°	228.89°
2 (m–i)	51.1	40°	0.775	0.379	−3.257	217.19°	228.50°
3 (e–i)	72.8	49°	0.841	0.324	−3.574	169.79°	211.11°

between two plates composed of different materials (glass and glass coated with polymer film) with free BCs and capillary parameters taken from [3]. The parameters of the interfaces are given in Table below Fig. 4. The volumes of liquids confined inside interfaces read $V_m = 0.4377$, $V_i = 0.8940$.

4. Conclusion

We formulate a variational problem for coexistence of axisymmetric interfaces of three immiscible liquids: two of them, i and m, immersed in a third liquid (or gas) e and trapped between two smooth solid bodies with axisymmetric surfaces S_1, S_2 and free contact lines. Assuming the volume constraints of two liquids i and m, we find the governing (Young–Laplace) equations (2.12) supplemented by boundary conditions and Young relation (2.13) on S_1, S_2 and transversality relations (2.16) on singular curve where all liquids meet together.

We consider two different cases when the problem allows the coexistence of five (Sect. 2) or three (Sect. 3) interfaces. In the first case, the problem is reduced to solving 16 boundary conditions, 4 Young relations and 4 transversality relations (2.19–2.23), i.e., 24 equations for 24 variables. In the second case, this number is reduced substantially, namely to 15 equations with 15 variables (3.5) including 10 boundary conditions, 3 Young relations and 2 transversality relations.

We derive the relationship (2.24) combining the constant mean curvatures of three different interfaces, e–m, m–i, e–i, and give consistency rules for interface coexistence (Sect. 2.1).

Another result is the vectorial Neumann triangle relation (2.32) at the triple point which is located on a singular curve. It has a clear physical interpretation as the balance equation of capillary forces. More importantly, it gives a new insight on the old assertion about the usual Young relations (2.30, 3.6) at a solid/liquid/gas interface refered by R. Finn [2] to T. Young: *the contact angle at a solid/liquid/gas interface is a physical constant depending only on the materials, and in no other way on the particular conditions of problem*, and a well-known contradiction with uncompensated normal force reaction of solid (see [2] and references therein). Indeed, being applied to the contact line of three continuous media, liquid–gas–solid, vectorial relation (2.32) assumes a non-smooth (singular) deformation [5] of solid surface with finite elastic modulus E.

Surface tension plays a negligible role at large scales, but it may be crucial when very flexible objects or microscopic sizes ℓ are considered. In the last decade, an interplay between surface forces of a liquid droplet and elasticity of a solid substrate was studied experimentally in slender structures of thickness h, where $\ell > \ell_{ec} = \sqrt{Eh^3/\gamma}$ and ℓ_{ec} denotes the elasto-capillary length: an elastic structure will be strongly deflected by surface tension forces (see [7] and references therein).

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